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CONTENTS

This issue of AVTOMATIKA I TELEMEKHANIKA contained an article commemorating the 90th anniversary of the birth of V. I. Lenin. Since that article did not discuss either the present state of Soviet research in automation or its historic development, and as a matter of fact, contained no information of scientific interest, it was omitted from the translation.

ANALYTICAL CONTROLLER DESIGN. I

A. M. Letov

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 21, No. 4, pp. 436-441, April, 1960
Original article submitted December 23, 1959

A solution is presented for the problem of analytical controller design in accordance with a given optimizing functional. The solution is given for the open region of definition of the system's differential equations. The cases of closed regions are worked out in Parts II and III of the present work, which are to be published in succeeding numbers of this journal.

1. Two Classes of Problems in Optimal System Theory

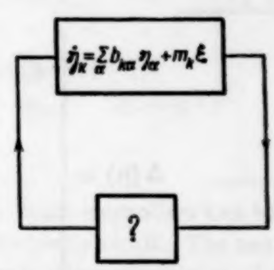
The construction of optimal systems leads to the solution of the allied mathematical problems, which are divided into two classes. The first of these is comprised of those problems which are related to the determination and calculation of the undisturbed modes of motion. Here, one has to do with those programs of automatic control for which the given undisturbed motion possesses the requisite extremal properties. We shall call such systems optimal with respect to control mode.

Systems for which the concept of optimality connotes the arrival at a given state in minimal time we shall find it convenient to call brachistochrone.

In the other class of problems fall those controllers which guarantee the existence of given properties of the disturbed motions (transient processes).

We shall call such systems optimal with respect to transient response. The set consisting of the object of control, the optimal programming element, and the optimal controller comprises an optimal automatic control system. In both cases the optimization problem can be treated as a two-point boundary problem, for the solution of which one may apply all the methods of the variational calculus and, in particular, the latest methods, developed in [1-7]. However, certain differences do exist between these two classes of problems. Thus, in the first case, the problem's solution is obtained in the form of a known function of time, which then becomes the basis for the design of the programming elements. In the second case, the analytical form of the control law appears as some function of the original coordinates of the control system, i.e., the problem consists of the construction of the controller's differential equation. Of course, the process of analytic controller construction is nothing else than the synthesis process. In particular, the synthesis process is here given the same meaning as in [1-5]. However, we choose not to use this term since many other authors use it with differing interpretations. On the other hand, the equivalent terminology "analytic construction" must suggest itself more strongly to engineers, since this terminology implies merely the search, by means of the tools of mathematical analysis, for the form of the controller's differential equation which will answer to the accepted criterion of optimality. In the present paper we consider the problem of the analytical construction (design) of optimal systems which are defined in open regions. In the succeeding papers we shall deal with the same problem as formulated for closed regions.

2. Initial Posing of the Analytical Design Problem



We consider a closed controlled system (cf. Fig. 1) in which the object's disturbed motion is given by the set of equations

$$g_k = \dot{\eta}_k - \Big(\sum_\alpha b_{k\alpha}\eta_\alpha + m_k\xi\Big) = 0 \qquad (2.1)$$

$$(k = 1, \ldots, n)$$

in the generalized coordinates η_k, while the equation of the controller for the coordinate ξ remains unknown. For definiteness, we shall assume that all the $b_{k\alpha}$ and m_k are given constants on which no restraints have as yet been imposed.

Let N be the open region in which (2.1) are given; any conditions in N of the form

$$\eta_1(0) = \eta_{10}, \ldots, \eta_n(0) = \eta_{n0}, \quad \xi(0) = \xi_0, \qquad (2.2)$$

$$\eta_1(\infty) = \ldots = \eta_n(\infty) = \xi(\infty) = 0$$

we shall call natural boundary conditions of the problem. They mean simply this, that whatever the transient response which arises in N, it must terminate, for t = ∞, with the system being found at the origin of coordinates.

As the criterion for system optimality, we choose the integral

$$I(\xi) = \int_0^\infty V\,dt \qquad (2.3)$$

of the positive definite quadratic form

$$V = \sum_k a_k \eta_k^2 + c\xi^2. \tag{2.4}$$

We shall seek such continuous functions $\xi, \eta_1 \ldots \eta_n$ (class C_1) with continuous first derivatives which give the integral in (2.3) a minimum value. The existence and uniqueness of the solution of this problem were proven in [7]. We now give ourselves the task of determining the form of this solution as well as the possibilities of using it for the purpose of an analytic design. The integral $I(\xi)$ is a functional defined on class C_1 of functions, and its value characterizes the integral of the squared error, weighted by the constants a_k and c, which the system will have in the course of the transient response lasting until $t = \infty$. The problem consists in writing in analytic form

$$F(\dot{\xi}, \xi, \eta_1, \ldots, \eta_n) = 0 \tag{2.5}$$

the control law which, in conjunction with (2.1), will form a stable system and will guarantee the existence of a minimum for the integral in (2.3).

3. Solution of the Problem

We are dealing with the Lagrange variational problem, the procedures for the solution of which are very well known. Specifically, we first set up the function

$$H = V + \sum_k \lambda_k g_k, \tag{3.1}$$

where the λ_k are arbitrary multipliers.

Then,

$$\frac{\partial H}{\partial \dot{\eta}_k} = \lambda_k, \quad \frac{\partial H}{\partial \eta_k} = 2a_k\eta_k - \sum_\alpha \lambda_\alpha b_{\alpha k}; \quad \frac{\partial H}{\partial \dot{\xi}} = 0, \tag{3.2}$$

$$\frac{\partial H}{\partial \xi} = 2c\xi - \sum_\alpha m_\alpha \lambda_\alpha. \quad (3.2)$$

The equations of the variational problem have the form

$$\dot{\lambda}_k = -\sum_j b_{jk}\lambda_j + 2a_k\eta_k, \quad 0 = 2c\xi - \sum_j m_j\lambda_j. \tag{3.3}$$

To these, (2.1) should be adjoined.

The further steps in the procedure for analytic design (excluding singular cases) are as follows:

1. The equations of the variational problem must be solved jointly for the functions η_k and λ_k. Let

$$\Delta(\mu) = \begin{vmatrix} b_{11}-\mu, \ldots, & b_{1n}, & \frac{m_1^2}{2c}, \ldots, & \frac{m_1 m_2}{2c} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1}, \ldots, & b_{nn}-\mu, & \frac{m_n m_1}{2c}, \ldots, & \frac{m_n^2}{2c} \\ 2a_1, \ldots, & 0, & -b_{11}-\mu, \ldots, & -b_{n1} \\ \cdots & \cdots & \cdots & \cdots \\ 0, \ldots, & 2a_n, & -b_{1n}, \ldots, & -b_{nn}-\mu \end{vmatrix} \tag{3.4}$$

be the system's characteristic determinant. It is easily proven that, if $\mu_1, \ldots, \mu_n$ are simple roots of the equation $\Delta(\mu) = 0$, then the numbers $-\mu_1, \ldots, -\mu_n$ are also simple roots. We assume that $\mu_1, \ldots, \mu_n$ are simple roots, $\mathrm{Re}\,\mu_k \neq 0$, and the roots are numbered in accordance with the inequalities

$$\mathrm{Re}\,\mu_k < 0 \quad (k = 1, \ldots, n). \tag{3.5}$$

This latter step is always possible by virtue of the aforementioned property of determinant (3.4). The general solution of the equations of the problem will consist of a linear combination of exponential functions of the form $c_k e^{\mu_k t}$ and $c_{n+k} e^{-\mu_k t}$ ($k = 1, \ldots, n$), and will contain 2n arbitrary constants

$$c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n}.$$

2. By virtue of (2.2)* at infinity we set

$$c_{n+k} = 0 \; (k = 1, \ldots, n)$$

where the constants c_k can be chosen from the initial conditions. However, this operation is not necessary.

3. In the remaining 2n formulas which define the functions $\eta_1, \ldots, \eta_n, \lambda_1, \ldots, \lambda_n$, we eliminate the n functions of time $c_k e^{\mu_k t}$, as a result of which we find that

$$\lambda_j = \sum_\alpha \Delta_{j\alpha}\eta_\alpha \quad (j = 1, \ldots, n), \tag{3.6}$$

where the $\Delta_{j\alpha}$ are completely defined constants.

4. We substitute the values found for the λ_j in the last equation of (3.3), thus finding the controller's equation

$$\xi = p_1\eta_1 + p_2\eta_2 + \ldots + p_n\eta_n. \tag{3.7}$$

It remains to verify that (3.7), in conjunction with the initial equations (2.1), forms a stable† automatic control system. This verification is easily carried out. Indeed, the optimal system found has a general solution which is a linear combination of the exponential functions $c_k e^{\mu_k t}$ ($k = 1, \ldots, n$). Consequently, the numbers μ_k ($k = 1, \ldots, n$) are the roots of the characteristic equation of system (2.1), (3.7). We obtained ideal controller (3.7) with rigid feedback and with an infinitely fast servomotor [8]. Thus, we arrive at the conclusion that functionals like (2.3) lead to controllers which can not be realized.

4. Possible Generalizations of Functional (2.3)

We now consider an indefinite quadratic form such as

$$T = \sum_k \sum_j c_{kj} \dot{\eta}_k \dot{\eta}_j + \dot{\xi}^2 \tag{4.1}$$

and repeat the formulation of the Lagrange problem for the functional

$$I(\xi) = \int_0^\infty (T + V)\, dt.$$

The value of the functional $I(\xi)$ characterizes the integral of the squared position and velocity errors which the system will have in the course of the transient response lasting until $t^* = \infty$.

We remark that the solution of the problem may be simplified by eliminating the derivatives $\dot{\eta}_k$ in accordance with the original equations. It is reasonable, therefore, to consider first the simpler functional of the form

$$I(\xi) = \int_0^\infty (V + \dot{\xi}^2)\, dt. \tag{4.2}$$

The computation will differ from the foregoing only in that now the expression $\partial H / \partial \dot{\xi} = 2\ddot{\xi}$ and does not equal zero. Therefore, instead of the last equation of (3.3), we get

$$2\ddot{\xi} = 2c\xi - \sum_j m_j \lambda_j. \tag{4.3}$$

This equation, in conjunction with (2.1) and the first n equations of (3.3), solves the given problem. The solution procedure reduces to the implementation of the following operations:

1. The equations of the problem must be solved for the functions η_k, λ_k and ξ.

Let

$$\overline{\Delta}(\mu) = \begin{vmatrix} b_{11}-\mu, & \ldots, & b_{1n}, & 0, & \ldots, & 0, & m_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1}, & \ldots, & b_{nn}-\mu, & 0, & \ldots, & 0, & m_n \\ 2a_1, & \ldots, & 0, & -b_{11}-\mu, & \ldots, & -b_{n1}, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & \ldots, & 2a_n, & -b_{1n}, & \ldots, & -b_{nn}-\mu, & 0 \\ 0, & \ldots, & 0, & -m_1, & \ldots, & -m_n, & 2c-\mu^2 \end{vmatrix} \tag{4.4}$$

be the system's characteristic determinant. As in the case of (3.4), it possesses the property that $\overline{\Delta}(\mu) = \overline{\Delta}(-\mu)$. Therefore, the equation $\Delta(\mu) = 0$ will have the roots $\mu_1, \ldots, \mu_{n+1}$ (which we assume to be simple), with the property enunciated by (3.5), plus the roots $-\mu_1, \ldots, -\mu_{n+1}$. The system's general solution will consist of a linear combination of exponential functions of the form $c_k e^{\mu_k t}$ and $c_{n+1+k} e^{-\mu_k t}$ ($k = 1, \ldots, n+1$) and will contain $2n+2$ arbitrary constants.

2. By virtue of (2.2) at infinity we set $c_{n+1+k} = 0$ ($k = 1, \ldots, n+1$). The remaining constants can be determined from the initial conditions.

3. We now take the solution found for the function ξ. In the $n+2$ formulas found for the functions $\eta_1, \ldots, \eta_n$, ξ and $\dot{\xi}$, we eliminate the functions of time $c_1 e^{\mu_1 t}, \ldots, c_{n+1} e^{\mu_{n+1} t}$, as a result of which we find the controller's equation

$$\dot{\xi} = \sum_\alpha p_\alpha \eta_\alpha - r\xi. \tag{4.5}$$

Here, the p_α and r are completely defined constants, determined in the course of implementing the enumerated operations. In conjunction with (2.1), (4.5) forms a stable optimal system. The stability, as in the previous case, is a consequence of the fact that the numbers $\mu_1, \ldots, \mu_{n+1}$ are roots of the optimal system's characteristic equation.

In contradistinction to the solution of Section 3, we obtained a linear controller with a limited variable servomotor speed. Such controllers can be realized if the region N is sufficiently small. The analytic design procedure in the case when the motion of the system is in a closed region N, when the optimal solution can be found on the region's boundary, will be considered in the subsequent papers.

5. Examples

For a first-order system, the equations of the variational problem will be

$$\dot{\eta} = b\eta + m\xi, \quad \dot{\lambda} = -b\lambda + 2a\eta, \quad 2\ddot{\xi} = 2c\xi - m\lambda. \tag{5.1}$$

To these there correspond the characteristic equation

$$\mu^4 - (b^2 + c)\mu^2 + am^2 + cb^2 = 0 \tag{5.2}$$

and the solution

$$\begin{gathered} \eta = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}, \\ \xi = \frac{\mu_1 - b}{m} c_1 e^{\mu_1 t} + \frac{\mu_2 - b}{m} c_2 e^{\mu_2 t}, \\ \dot{\xi} = \frac{\mu_1(\mu_1 - b)}{m} c_1 e^{\mu_1 t} + \frac{\mu_2(\mu_2 - b)}{m} c_2 e^{\mu_2 t}. \end{gathered} \tag{5.3}$$

The controller's equation has the form

$$\begin{vmatrix} 1 & 1 & \eta \\ \dfrac{\mu_1 - b}{m} & \dfrac{\mu_2 - b}{m} & \xi \\ \dfrac{\mu_1(\mu_1 - b)}{m} & \dfrac{\mu_2(\mu_2 - b)}{m} & \dot{\xi} \end{vmatrix} = 0$$

or, in expanded form,

$$\dot{\xi} = -\frac{(\mu_1 - b)(\mu_2 - b)}{m}\eta + (\mu_1 + \mu_2 - b)\xi. \quad (5.4)$$

For a second example, we consider the second-order system

$$\dot{\eta}_1 = b_{11}\eta_1 + b_{12}\eta_2 + m_1\xi, \quad \dot{\eta}_2 = b_{21}\eta_1 + b_{22}\eta_2 + m_2\xi. \quad (5.5)$$

To obtain the equations of the variational problem, we must append

$$\begin{aligned} \dot{\lambda}_1 &= -b_{11}\lambda_1 - b_{21}\lambda_2 + 2a_1\eta_1, \\ \dot{\lambda}_2 &= -b_{12}\lambda_1 - b_{22}\lambda_2 + 2a_2\eta_2, \\ 2\ddot{\xi} &= 2c\xi - m_1\lambda_1 - m_2\lambda_2. \end{aligned} \quad (5.6)$$

The characteristic equation has the form

$$\overline{\Delta}(\mu) = \begin{vmatrix} b_{11} - \mu, & b_{12} & 0, & 0, & m_1 \\ b_{21}, & b_{22} - \mu, & 0, & 0, & m_2 \\ 2a_1 & 0, & -b_{11} - \mu, & -b_{21}, & 0 \\ 0, & 2a_2, & -b_{12}, & -b_{22} - \mu, & 0 \\ 0, & 0, & -m_1, & -m_2 & 2c - 2\mu^2 \end{vmatrix} = 0.$$

Let μ_1, μ_2, and μ_3 be the three roots for which $\mathrm{Re}\,\mu_s < 0$. We then have

$$\eta_1 = \sum_{s=1}^{3} \Delta_1(\mu_s) e^{\mu_s t} c_s, \quad \eta_2 = \sum_{s=1}^{3} \Delta_2(\mu_s) e^{\mu_s t} c_s,$$
$$\xi = \sum_{s=1}^{3} \Delta_5(\mu_s) e^{\mu_s t} c_s, \quad \dot{\xi} = \sum_{s=1}^{3} \mu_s \Delta_5(\mu_s) e^{\mu_s t} c_s. \quad (5.8)$$

Here, Δ_1, Δ_2, and Δ_5 are the minors of the first, second, and fifth elements of the first row of the determinant in (5.7). It is obvious that the equation sought for the optimal controller will have the form

$$\begin{vmatrix} \Delta_1(\mu_1) & \Delta_1(\mu_2) & \Delta_1(\mu_3) & \eta_1 \\ \Delta_2(\mu_1) & \Delta_2(\mu_2) & \Delta_2(\mu_3) & \eta_2 \\ \Delta_5(\mu_1) & \Delta_5(\mu_2) & \Delta_5(\mu_3) & \xi \\ \mu_1\Delta_5(\mu_1) & \mu_2\Delta_5(\mu_2) & \mu_3\Delta_5(\mu_3) & \dot{\xi} \end{vmatrix} = 0. \quad (5.9)$$

The author sincerely thanks E. A. Barbashin and N. N. Krasovskii for their active participation in the discussions of the present problem, and for their helpful suggestions.

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*For the given case, the boundary condition for the function ξ is superfluous.

† The author wishes to thank M. A. Aizerman for the advice which permitted him to avoid an erroneous judgment on this property, which was contained in the first version of the paper.

‡ See English translation.

APPLICATION OF THE KRYLOV – BOGOLYUBOV METHOD OF CONSTRUCTING ASYMPTOTIC APPROXIMATIONS TO THE INVESTIGATION OF SYSTEMS WITH LAGS*

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Original article submitted July 4, 1959

The present paper treats of the application of the Krylov-Bogolyubov method of constructing asymptotic approximations to the investigation of linear and nonlinear systems with lags. The effectiveness of this application of the method is illustrated by the example of the investigation of the Minorsky equation.

Introduction

Today, as an effective instrument for the approximate investigation of nonlinear instruments, one may use the widely disseminated method developed by N. M. Krylov and N. N. Bogolyubov [1] for the construction of asymptotic approximations. This method, in a simplified variant (when only first approximations are considered), is called the harmonic balance method [2].

The nub of the asymptotic method of Krylov and Bogolyubov as applied, for example, to a nonlinear differential equation of the form

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}), \tag{1}$$

where ϵ is a small positive parameter, reduces to the search for the general solution of this equation in the form of a formal series expansion in powers of the small parameter ϵ:

$$x = A\cos\psi + \varepsilon u_1(A, \psi) + \varepsilon^2 u_2(A, \psi) + \ldots, \tag{2}$$

in which $u_1(A, \psi)$, $u_2(A, \psi)$,.... are periodic functions with period 2π. An additional condition is imposed on the functions u_1, u_2,..., which is expressed by the absence of the first harmonics in the Fourier expansions of these functions.

As a first approximation to the solution of (1), we take the expression

$$x = A\cos\psi, \tag{3}$$

in which the amplitude A and the phase angle ψ are defined by the equations

$$\dot{A} = \varepsilon D_1(A), \quad \dot{\psi} = \omega + \varepsilon B_1(A). \tag{4}$$

The coefficients $D_1(A)$ and $B_1(A)$ in (4) are defined as the first (fundamental) harmonics of the expansion of the nonlinear function in a Fourier series:

$$D_1(A) = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(A\cos\psi, -A\omega\sin\psi)\sin\psi\, d\psi, \tag{5}$$

$$B_1(A) = -\frac{1}{2\pi\omega A}\int_0^{2\pi} f(A\cos\psi, -A\omega\sin\psi)\cos\psi\, d\psi. \tag{6}$$

As a second approximation to the solution of (1), we take the expression

$$x = A\cos\psi + \varepsilon u_1(A, \psi), \tag{7}$$

in which A and ψ are defined by the equations

$$\dot{A} = \varepsilon D_1(A) + \varepsilon^2 D_2(A),$$
$$\dot{\psi} = \omega + \varepsilon B_1(A) + \varepsilon^2 B_2(A), \tag{8}$$

and u_1 is defined by the formula

$$u_1 = -\frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(A)\cos n\psi + h_n(A)\sin n\psi}{n^2 - 1}, \tag{9}$$

where $g_n(A)$ and $h_n(A)$ are coefficients of the Fourier series:

$$g_n(A) = \frac{1}{2\pi}\int_0^{2\pi} f(A\cos\psi, -A\omega\sin\psi)\cos n\psi\, d\psi,$$
$$h_n(A) = \frac{1}{2\pi}\int_0^{2\pi} f(A\cos\psi, -A\omega\sin\psi)\sin n\psi\, d\psi, \tag{10}$$

and the coefficients $D_2(A)$ and $B_2(A)$ in (8) are defined by the corresponding formulas (cf. [2]).

As a third approximation to the solution of (1) we take the expression

$$x = A\cos\psi + \varepsilon u_1(A, \psi) + \varepsilon^2 u_2(A, \psi) \text{ etc.} \tag{11}$$

In automatic control systems (ACS) one is frequently most interested in the so-called stationary (steady-state) oscillations. This is related to the fact that all nonstationary oscillations, after a quite rapid transient response, approxi-

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***Some results of this paper were briefly presented at the Sixth Scientific-Engineering Conference of Junior Scientists on Automatic Control of the IAT AN SSSR (Jan. 19-21, 1959) and, synoptically, at a seminar on differential-difference equations at Moscow State University (May 28, 1959).**

mate to steady-state oscillations and, almost immediately after the initiation of an oscillatory process, this process can be considered as being stationary. A stationary mode, as is well known, is characterized by a constant amplitude, i.e., such that the derivative of the amplitude with respect to time is zero. For the system described by (1), the steady state is defined by the condition

$$\dot{A} = \varepsilon D_1(A) + \varepsilon^2 D_2(A) + \ldots = 0. \qquad (12)$$

The method under consideration, i.e., the Krylov-Bogolyubov method of constructing asymptotic approximations, can be successfully employed, as will be shown below, for differential equations with lagging arguments. With this, the Krylov-Bogolyubov method, when combined with the Routh-Hurwitz criterion, turns out to be a simple and effective means both for finding the steady-state periodic mode and for determining the boundary of the region of stability of the controlled system with lags described by linear and quasi-linear differential equations with lagging arguments. The present paper is based on [3], in which there is contained the foundation for the method of approximate computations of periodic solutions of differential equations with lagging arguments, as well as on the results of simulation studies. The foundation for the possibility of using the asymptotic methods of Krylov and Boglyubov for the investigation of systems with lags will be given in the second part of the present paper.

1. Equivalent Linearization of Nonlinear Oscillatory ACS with Lags by the Krylov-Bogolyubov Method

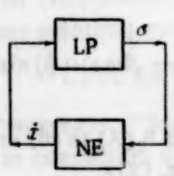

Fig. 1. Block schematic of an ACS with lags. LP is the linear portion and NE is the nonlinear element with lags.

We consider an ACS consisting, in the general case, of a linear portion and a nonlinear element with lags (Fig. 1). The equation of the linear portion may, in general, be arbitrary† while the equation of the nonlinear element has the form

$$\dot{x}(t) = f[\sigma(t-\tau)]. \qquad (1.1)$$

An equation of the form of (1.1), in which the lag enters into the argument of the nonlinear function, may be treated, in particular cases, as the equation describing the executive organ of an ACS. In this case, the function $f[\sigma(t-\tau)]$ represents the acting generalized lagged force engendered by the executive organ in accordance with the value of the argument $\sigma(t-\tau)$, where τ is a positive constant which characterizes the temporal delay. In each controlled system, the argument σ expresses the total (sum) pulse (signal) formed in accordance with the control law adopted for the problem. It is assumed that the function $f[\sigma(t-\tau)]$ lies in a subclass A' of class A of functions [4], i.e., satisfies conditions of the form

$$f[\sigma(t-\tau)] = 0 \quad \text{for} \quad \sigma(t-\tau) = 0;$$

$$\left[\frac{df[\sigma(t-\tau)]}{d\sigma(t-\tau)}\right]_{\sigma(t-\tau)=0} \geqslant k > 0,$$

$$\sigma(t-\tau)\varphi[\sigma(t-\tau)] > 0$$

$$\text{for} \quad \sigma(t-\tau) \neq 0,$$

where $\varphi[\sigma(t-\tau)] = f[\sigma(t-\tau)] - k\sigma(t-\tau)$, and k is a given constant. This function also satisfies the Dirichlet conditions in the control range $2|\bar{\sigma}|$, and can be presented in the form of a linear portion and a small nonlinear correction:

$$f[\sigma(t-\tau)] = k\sigma(t-\tau) + \varepsilon\varphi[\sigma(t-\tau)], \qquad (1.2)$$

where ϵ($\epsilon \ll 1$), introduced by means of the substitution $\bar{x}(t) = \epsilon\, x(t)$, is a small parameter.

We now find a linear element which is equivalent, in a first approximation, to the nonlinear element of (1.1). We write the differential equation of the linear element in the form

$$\dot{x}(t) = a_1(A)\sigma(t) + a_2(A)\dot{\sigma}(t), \qquad (1.3)$$

where $a_1(A)$ and $a_2(A)$ are the amplitude-dependent coefficients of the linear element.

It is required to determine the values of the coefficients $a_1(A)$ and $a_2(A)$ and to resolve the question of the physical interpretation of equivalent formula (1.3). For this we can use the standard electrical engineering concepts of active and reactive power. The active power developed by the executive organ during one period of oscillation corresponding, in the given case, to average power, will equal the work of the force $f[\sigma(t-\tau)]$ during a period T of oscillation, divided by the magnitude of T, i.e.,

$$\frac{1}{T}\int_0^T f[\sigma(t-\tau)]\,\dot{\sigma}(t)\,dt. \qquad (1.4)$$

The reactive power will equal

$$\frac{1}{T}\int_0^T f[\sigma(t-\tau)]\,\dot{\sigma}\left(t-\frac{T}{4}\right)dt. \qquad (1.5)$$

We now show that the value of the coefficient $a_1(A)$ can be obtained by equating the reactive powers of the nonlinear and the linear elements, and that the value of the coefficient $a_2(A)$ can be obtained by a corresponding equating of the active powers. Thus, in accordance with (1.5) and presenting $f[\sigma(t-\tau)]$ in the form of (1.2), we obtain

† The Krylov-Bogolyubov method permits the linearization of a system's nonlinear element independently of the other system parameters.

$$\frac{1}{T}\int_0^T f[\sigma(t-\tau)]\,\dot{\sigma}\left(t-\frac{T}{4}\right)dt =$$

$$= \frac{k}{T}\int_0^T \sigma(t-\tau)\,\dot{\sigma}\left(t-\frac{T}{4}\right)dt +$$

$$+\frac{\varepsilon}{T}\int_0^T \varphi[\sigma(t-\tau)]\,\dot{\sigma}\left(t-\frac{T}{4}\right)dt. \qquad (1.6)$$

We assume, in a first approximation, that the quantity $\sigma(t)$, during the time interval $T = 2\pi/\omega$, varies harmonically:

$$\sigma(t) = A\cos\omega t, \quad \dot{\sigma}(t) = -A\omega\sin\omega t. \qquad (1.7)$$

By substituting the quantity in (1.7) into (1.6) for the reactive power, and introducing the new independent variable $\psi = \omega t$, we get

$$\frac{1}{T}\int_0^T f[\sigma(t-\tau)]\,\dot{\sigma}\left(t-\frac{T}{4}\right)dt = \frac{A^2k\omega}{2}\cos\omega\tau +$$

$$+\frac{\varepsilon A\omega}{2\pi}\int_0^{2\pi}\varphi[A\cos\omega(t-\tau)]\cos\psi\,d\psi. \qquad (1.8)$$

By substituting the quantity defined by (1.3) into (1.5) for the reactive power, and by taking (1.7) into account, we have

$$\frac{1}{T}\int_0^T a_1(A)\,\sigma(t)\,\dot{\sigma}\left(t-\frac{T}{4}\right)dt -$$

$$-\frac{1}{T}\int_0^T a_2(A)\,\dot{\sigma}(t)\,\dot{\sigma}\left(t-\frac{T}{4}\right)dt = \frac{a_1(A)A^2\omega}{2}. \qquad (1.9)$$

From a comparison of (1.8) and (1.9) we obtain the value of coefficient $a_1(A)$:

$$a_1(A) = k\cos\omega\tau - \frac{\varepsilon}{\pi A}\int_0^{2\pi}\varphi[A\cos\omega(t-\tau)]\cos\psi\,d\psi. \qquad (1.10)$$

In exactly the same fashion, by substituting $f[\sigma(t-\tau)]$, in the form of (1.2), in (1.4) for the active power, we get

$$\frac{1}{T}\int_0^T f[\sigma(t-\tau)]\,\dot{\sigma}(t)\,dt = \qquad (1.11)$$

$$= \frac{k}{T}\int_0^T \sigma(t-\tau)\,\dot{\sigma}(t)\,dt + \frac{\varepsilon}{T}\int_0^T \varphi[\sigma(t-\tau)]\,\dot{\sigma}(t)\,dt.$$

By substituting the quantity (1.7) in (1.11), we obtain

$$\frac{1}{T}\int_0^T f[\sigma(t-\tau)]\,\dot{\sigma}(t)\,dt = -\frac{kA^2\omega\sin\omega\tau}{2} -$$

$$-\frac{\varepsilon A\omega}{2\pi}\int_0^{2\pi}\varphi[A\cos\omega(t-\tau)]\sin\psi\,d\psi. \qquad (1.12)$$

By substituting in (1.4) for the active power, the quantity defined by (1.3), and by taking (1.7) into account, we have

$$\frac{1}{T}\int_0^T a_1(A)\,\sigma(t)\,\dot{\sigma}(t)\,dt + \frac{1}{T}\int_0^T a_2(A)\,\dot{\sigma}^2(t)\,dt =$$

$$= \frac{A^2\omega^2 a_2(A)}{2}. \qquad (1.13)$$

From a comparison of (1.12) and (1.13) we obtain the value of coefficient $a_2(A)$:

$$a_2(A) = -\frac{k\sin\omega\tau}{\omega} - \frac{\varepsilon}{\pi A\omega}\int_0^{2\pi}\varphi[A\cos\omega(t-\tau)]\sin\psi\,d\psi. \qquad (1.14)$$

The method of determining the equivalent coefficients $a_1(A)$ and $a_2(A)$ by (1.10) and (1.14) can be called the power principle, or the energetic balance principle for a nonlinear element with lags. It is easily shown (cf. the Appendix) that, when the harmonic balance principle is applied in the case when $n = 1$, one can obtain the values themselves of the coefficients. In accordance with (A.8) and (A.9), (1.10) and (1.14) can be given in the form

$$a_1(A) = \left(k + \frac{\varepsilon g_1(A)}{A}\right)\cos\omega\tau - \frac{\varepsilon h_1(A)}{A}\sin\omega\tau, \qquad (1.15)$$

$$a_2(A) = -\left(k + \frac{\varepsilon g_1(A)}{A}\right)\frac{\sin\omega\tau}{\omega} - \frac{\varepsilon h_1(A)}{A\omega}\cos\omega\tau, \qquad (1.16)$$

where $g_1(A)$ and $h_1(A)$ are the coefficients of the nonlinear function's Fourier series expansion for the case $n = 1$.

2. Determination of the Steady-State Periodic Mode Parameters in Systems with Time Lags

As the first example of the use of the Krylov-Bogolyubov method of asymptotic approximations, we consider a system described by an ordinary first-order linear differential equation with constant coefficients and lagging argument:

$$\dot{x} + kx(t-\tau) = 0. \qquad (2.1)$$

Equation (2.1) was investigated by Schmidt [5] by means of a rigorous method for extracting the roots of the so-called transcendental function‡. Schmidt showed that (2.1) possesses a periodic solution of the form

$$x = c_1\cos\omega t + c_2\sin\omega t, \qquad (2.2)$$

where c_1 and c_2 are arbitrary constants, with the condition that ω is defined by means of the equation

$$\omega = k = \frac{1}{\tau}\left(\frac{\pi}{2} \pm 2\pi n\right) \quad (n = 0, 1, 2, \ldots) \qquad (2.3)$$

‡E. Schmidt considered the transcendental function

$$l(\omega) = i\omega - k\exp(-i\omega\tau) = -k\cos\omega\tau + i(\omega + k\sin\omega\tau). \qquad (2.1a)$$

In his notation, instead of k, τ, ω in (2.1) and (2.1a), there stand λ, h, ν.

By the Krylov-Bogolyubov method, the coefficients a_1 and a_2 of the equivalent system, defined by (1.15) and (1.16), will equal for the example under consideration, ($\epsilon = 0$):

$$ {}_1 = k \cos \omega\tau, \quad a_2 = \frac{-k \sin \omega\tau}{\omega}, \tag{2.4}$$

and the system's equivalent equation has the form

$$\left(1 - \frac{k \sin \omega\tau}{\omega}\right)\dot{x} + k \cos \omega\tau\, x = 0. \tag{2.5}$$

A periodic solution of (2.5) occurs with the conditions that

$$1 - \frac{k \sin \omega\tau}{\omega} = 0, \qquad k \cos \omega\tau = 0. \tag{2.6}$$

As is easily seen, (2.3) follows from this. Thus, the first approximation gives a result which coincides quantitatively with the exact solution of the ordinary linear differential equation with lagged argument which is being considered.

As a second example of the use of the Krylov-Bogolyubov method, we consider the questions connected with the search for the steady-state periodic mode (autooscillation) which occurs under certain conditions in nonlinear oscillatory systems with artifically introduced damping. The system is described by the so-called Minorsky** differential equation

$$\ddot{y}(t) + 2r\dot{y}(t) + \omega_0^2 y(t) + 2q\dot{y}(t-\tau) = \rho\dot{y}^3(t-\tau). \tag{2.7}$$

where $\underline{r}$, ω_0 and $\underline{q}$ are positive constants and ρ is a small positive constant. By considering ρ to be sufficiently small, we set

$$\rho = \varepsilon\nu$$

and present (2.7) in the form of a system of two equations:

$$\dot{y} = z. \quad \dot{z} = -2rz - 2qz(t-\tau) - \omega_0^2 y + \varepsilon\nu z^3(t-\tau). \tag{2.8}$$

On the basis of the theorem on existence and uniqueness proven by Krasovskii†† [3], one can assert that, if the quasi-linear system (2.8) has a periodic solution for ϵ=0 (the so-called generating solution) y(t,0) then, if the condition $|\epsilon| < \epsilon_0$ holds, where ϵ_0 is some positive constant, there exists a unique periodic solution $y(t, \epsilon)$.

We start initially with the periodic solution for $\epsilon = 0$ (the generating solution), defined by the system of equations

$$\dot{y} = z. \quad \dot{z} = -2rz - 2qz(t-\tau) - \omega_0^2 y. \tag{2.9}$$

By defining the coefficients a_1 and a_2 of the equivalent system by (1.15) and (1.16) (for the case $\epsilon = 0$), we obtain the following equality:

$$2qz(t-\tau) = a_1 z(t) + a_2\dot{z}(t) = \tag{2.10}$$

$$= 2q\cos\omega\tau z(t) - \frac{2q \sin \omega\tau}{\omega} z(t).$$

By substituting the value of $2qz(t-\tau)$ in (2.9), we get

$$\dot{y} = z. \quad \dot{z} = -b_1 z - b_2 y. \tag{2.11}$$

where

$$b_1 = \frac{2r + 2q\cos\omega\tau}{1 - \frac{2q\sin\omega\tau}{\omega}}, \quad b_2 = \frac{\omega_0^2}{1 - \frac{2q\sin\omega\tau}{\omega}}.$$

The characteristic equation of (2.11) has the form

$$D(\lambda) = \begin{vmatrix} -\lambda & +1 \\ -b_2 & -b_1 - \lambda \end{vmatrix} = 0 \tag{2.12}$$

or

$$\lambda^2 + b_1\lambda + b_2 = 0. \tag{2.13}$$

Equation (2.13) is an ordinary algebraic equation, and one may apply to it the familiar classical formulas of the theory of linear oscillations, in particular, the Routh-Hurwitz stability criterion. By this criterion, the system described by characteristic (2.13) will lie on the boundary of stability and will manifest undamped periodic oscillations in case

$$R = \Delta_{n-1} = b_1 = 0, \tag{2.14}$$

where Δ_{n-1} is the (n-1)th Hurwitz determinant.

Condition (2.14) occurs when

$$R = 2r + 2q\cos\omega\tau = 0. \tag{2.15}$$

The frequency of the periodic oscillations is defined by the formula

$$\omega^2 = \frac{\omega_0^2}{1 - \frac{2q\sin\omega\tau}{\omega}}. \tag{2.16}$$

By eliminating $\underline{q}$ from (2.15) and (2.16), we find that

$$\operatorname{tg}\omega\tau = -\frac{\omega^2 - \omega_0^2}{2r\omega}. \tag{2.17}$$

It is easily seen that the periodic solution of (2.13) occurs under the following conditions:

$$\operatorname{tg}\omega\tau < 0. \quad \cos\omega\tau < 0 \tag{2.18}$$

or

$$\frac{\pi}{2} + 2\pi n \leqslant \omega\tau < \pi + 2\pi n \qquad (n = 0, 1, 2, \ldots) \tag{2.19}$$

In case of equality, $\omega\tau = \pi + 2\pi n$, the frequency ω of the periodic oscillations either coincides with, or is a multiple of, the natural frequency ω_0 of the system, and the phenomenon of resonance supervenes.

The frequency of the periodic oscillations is determined by a graphic solution of transcendental (2.17), and must satisfy (2.19). The amplitude A is an arbitrary constant, as is ordinarily the case when linear systems are being considered. It should be mentioned that, in the example under consideration, an arbitrarily small delay $\tau > 0$ may give rise to periodic oscillations in a system described by (2.9). In the case when $\tau = 0$, no periodic oscillations exist in the system described by (2.9) for any combination of the coefficients since, by our assumptions, the coeffi-

** An equation of the form of (2.7) was investigated by Minorsky [6,7], and then by Pinney [9], as related to the search for periodic oscillations.

†† Independently of Krasovskii, an analogous theorem was proven by Khalanai [10].

cients ω_0, r, q are positive nonzero numbers.

We now determine the periodic solution of (2.8) for $\epsilon \neq 0$. As a first approximation, we give the periodic solution in the form

$$y = A\cos\omega t, \tag{2.20}$$

where the amplitude A and the frequency ω are determined from the solution of the equation obtained from (2.8) by use of the Krylov-Bogolyubov method of equivalent linearization. Determining the coefficients of the equivalent system from (1.15) and (1.16), we have the following equations:

$$\rho z^3(t-\tau) = a_3 z(t) + a_4\dot{z}(t) = \frac{3}{4}\rho A^2\cos\omega\tau z(t) - \frac{3}{4}\rho A^2\frac{\sin\omega\tau}{\omega}\dot{z}(t). \tag{2.21}$$

By substituting the values of $2qz(t-\tau)$ and $\rho z^3(t-\tau)$ in (2.8), we get

$$\dot{y} = z,\quad \dot{z} = -b_3 z - b_4 y. \tag{2.22}$$

where

$$b_3 = \frac{2r + \cos\omega\tau\left(2q - \frac{3}{4}\rho A^2\right)}{1 - \frac{\sin\omega\tau}{\omega}\left(2q - \frac{3}{4}\rho A^2\right)}.$$

$$b_4 = \frac{\omega_0^2}{1 - \frac{\sin\omega\tau}{\omega}\left(2q - \frac{3}{4}\rho A^2\right)}.$$

In this case, the system's characteristic equation will be

$$D(\lambda) = \begin{vmatrix} -\lambda & +1 \\ -b_4 & -b_3-\lambda \end{vmatrix} = 0 \tag{2.23}$$

or

$$\lambda^2 + b_3\lambda + b_4 = 0. \tag{2.24}$$

For fixed values of the amplitude A, characteristic equation (2.24) is an ordinary algebraic equation, to which we apply the Routh-Hurwitz stability criterion. By this criterion, the system described by (2.24) will lie on the boundary of stability, and periodic oscillations will occur in it if

$$R(A) = \Delta_{n-1} = b_3 = 0. \tag{2.25}$$

which occurs for

$$R(A) = 2r + \cos\omega\tau\left(2q - \frac{3}{4}\rho A^2\right) = 0. \tag{2.26}$$

With this, the frequency of the periodic oscillations is determined by the formula

$$\omega^2 = \frac{\omega_0^2}{1 - \frac{\sin\omega\tau}{\omega}\left(2q - \frac{3}{4}\rho A^2\right)}. \tag{2.27}$$

By eliminating $2q - 3\rho A^2/4$ from (2.26) and (2.27), we find

$$\operatorname{tg}\omega\tau = -\frac{\omega^2 - \omega_0^2}{2r\omega}. \tag{2.28}$$

The expression for the amplitude is determined from (2.26):

$$A = \left[\frac{8(r + q\cos\omega\tau)}{3\rho\cos\omega\tau}\right]^{1/2}. \tag{2.29}$$

Equations (2.28) and (2.29) must satisfy (2.19) and, moreover, the condition

$$q\,|\cos\omega\tau| > r \qquad \left(\text{for } q > \frac{3}{8}\rho A^2\right). \tag{2.30}$$

which, if satisfied, makes (2.29) physically meaningful (A is real).

The joint solution of (2.28) and (2.29) is most easily begun with a determination of the frequency ω of the steady-state periodic oscillations, specifically, with a graphic solution of transcendental equation (2.28). For this, we present (2.28) in the form of two terms:

$$\operatorname{tg}\omega\tau = \frac{\omega_0^2}{2r\omega} - \frac{1}{2r}\omega. \tag{2.31}$$

the first term of which is the equilateral hyperbola $\xi_1 = \omega_0^2/2r\omega$ with the ξ and ω axes for asymptotes, and the second term is the straight line $\xi_2 = \omega/2r$, with slope $-1/2r$. Using algebraic addition of the ordinates of these two curves for fixed values of ω_0 and r, we obtain the sum curve (Fig. 2), whose intersections with the tangent curve in the second, sixth, etc. quadrants define the frequencies of the steady-state oscillations of which, if the graph is to be believed, there is an infinite set. However, it is easily shown that not all, but only a strictly finite number n, of the frequencies obtained from the graphic solution of (2.28) satisfy (2.30) and, consequently, (2.29). Indeed, it is clear from Fig. 2 that, as the ω_i tend to infinity ($i = 2,3,\ldots,n,\ldots$), where we understand by ω_i the frequencies obtained from the intersections of the tangent curves with the curve in the sixth, tenth, etc. quadrants, $\tan\omega_i\tau \to \infty$. With this, $\cos\omega_i\tau \to 0$ and, for any frequency ω_{n+1}, there can occur a violation of (2.30), i.e., the condition of physical realizability of (2.29) no longer holds. Thus, as a first approximation to the solution of (2.7), we obtain the expression‡‡

$$y(t) = \sum_{s=1}^{n} A_s\cos\omega_s t. \tag{2.32}$$

in which the amplitudes A_s ($s = 1,2,\ldots,n$) are determined from (2.29) and the frequencies corresponding to them, which satisfy (2.28) and (2.29), are determined from the graphic solution of (2.28).

As a second approximation to the solution of (2.7), neglecting quantities of the order of smallness of ρ^2, we get the expression

$$y(t) = \sum_{s=1}^{n} A_s\cos\omega_s t + \rho\sum_{s=1}^{n} u_1(A_s, \psi_s). \tag{2.33}$$

where u_1 is defined by a formula analogous to (9):

$$u_1 = -\frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{\bar{g}_n(A)\cos n\psi + \bar{h}_n(A)\sin n\psi}{n^2-1}. \tag{2.34}$$

‡‡The proof that every solution y(t) can be given in the form of a sum of partial solutions is carried out in the second part of this paper.

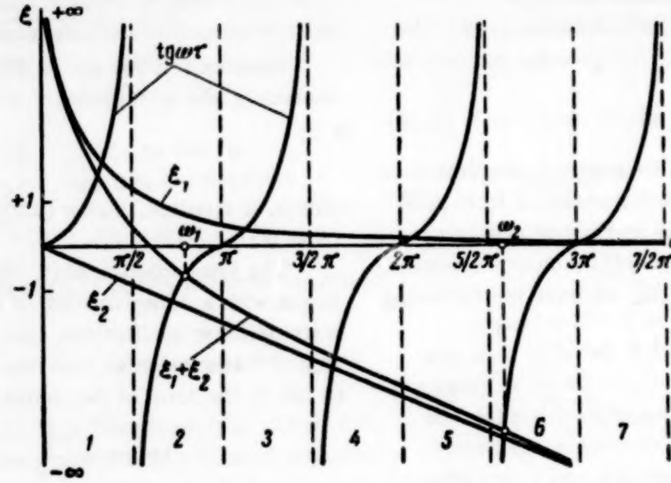

Fig. 2. Graphic solution of transcendental equation (2.28).

and $\bar{g}_n(A)$ and $\bar{h}_n(A)$ are defined by (A.8) and (A.9). In our case,

$$u_1 = \frac{1}{32} A^3 \frac{\cos 3\omega (t-\tau)}{\omega^2}. \tag{2.35}$$

Then, the second approximation has the form

$$y(t) = \sum_{s=1}^{n} A_s \cos \omega_s t + \frac{\rho}{32} \sum_{s=1}^{n} \frac{A_s^3 \cos 3\omega_s (t-\tau)}{\omega_s^2}$$

etc. (2.36)

3. Construction of the Boundaries of the Stability Region for the System of Equations (2.22) and the Investigation of Stability of the Steady-State Periodic Mode (Auto-Oscillation) ***

In our example, the boundary of the stability region, as was already stated, is defined by the condition

$$R(A) = 2r + \cos \omega\tau \left(2q - \frac{3}{4}\rho A^2\right) = 0 \quad (\cos \omega\tau < 0). \tag{3.1}$$

By introducing the notation $d(A) = q - 3\rho A^2/8$ and $k(\omega) = 1/|\cos \omega\tau|$, we construct the boundary of the stability region in the r, d(A) plane. This boundary is some curve whose equation may be written in the form

$$d(A) = k(\omega) r. \tag{3.2}$$

For the fixed values $\omega_0^2 = 20$, $\tau = 0.1$ sec, the boundary of the stability region is shown in Fig. 3. It divides the r, d(A) plane into the region of stability, where $R(A) > 0$, and the region of instability, where $R(A) < 0$. Since $d(A)$ is a function of amplitude, each fixed value of A has its own representative point. Therefore, the oscillations corresponding to boundary points of the system's parameter values, defined by (3.1), may be stable for cer-

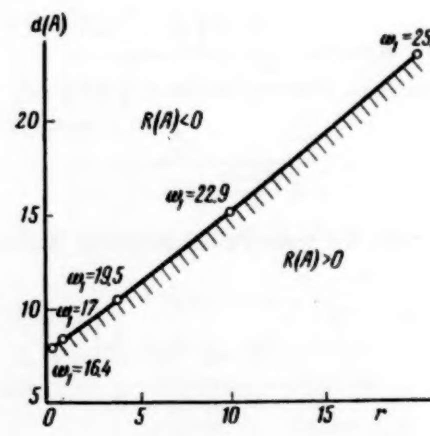

Fig. 3. Boundary of the stability region of system of equations (2.22) in the r, d(A) plane.

tain values of A and unstable for other amplitude values. The oscillations on the boundary of the region will be stable if

$$\left(\frac{dR(A)}{dA}\right)_{R(A)=0} > 0, \tag{3.3}$$

and unstable if this expression is negative.

In analogy with the criterion for determining safe and dangerous boundaries due to Bautin [11], we may assume that the boundary is dangerous "in the small" if the following inequality holds in its neighborhood:

$$\left(\frac{dR(A)}{dA}\right)_{A\to 0} < 0. \tag{3.4}$$

If this expression is positive, then the boundary is safe "in the small". Dangerous and safe portions of the boundary

***In investigating the stability of the steady-state oscillatory mode, we take the amplitude $A_1 = A$ of the fundamental harmonic into account, but ignore the amplitudes A_i ($i = 2,3,\ldots$) as being small high-frequency quantities.

are separated from each other by bifurcation points which are characterized by the condition

$$\left(\frac{dR(A)}{dA}\right)_{R(A)=0} = 0. \qquad (3.5)$$

Bautin's idea may easily be generalized to the case of stability "in the large", i.e., for $A \to \infty$. Mathematically, this may be determined as follows. The boundary of stability of the system is safe "in the large" if the following inequality holds:

$$\left(\frac{dR(A)}{dA}\right)_{A\to\infty} > 0. \qquad (3.6)$$

If the inverse inequality holds, then the boundary is dangerous "in the large".

Since, for nonlinear systems with lags R(A) depends not only on amplitude but also on the frequency of oscillation, then (3.3) can be rewritten in the form

$$\left(\frac{dR(A)}{dA}\right)_{R(A)=0} = \left(\frac{\partial R(A)}{\partial A}\right)_{R(A)=0} + \left(\frac{\partial R(A)}{\partial \omega}\right)_{R(A)=0} \frac{d\omega}{dA} > 0. \qquad (3.7)$$

For the example under consideration, we have

$$\left(\frac{dR(A)}{dA}\right)_{R(A)=0} = -\frac{3}{2} A\rho \cos\omega\tau - \tau \sin\omega\tau \left(2q - \frac{3}{4}\rho A^2\right)\frac{d\omega}{dA},$$

$$\frac{d\omega}{dA} = \frac{3\omega\rho A \sin\omega\tau}{2\left[\left(2q - \frac{3}{4}\rho A^2\right)(\sin\omega\tau + \omega\tau\cos\omega\tau) - 2\omega\right]}, \qquad (3.8)$$

where

$$\sin\omega\tau = \frac{\omega^2 - \omega_0^2}{\omega\left(2q - \frac{3}{4}\rho A^2\right)}, \qquad (3.9)$$

$$\cos\omega\tau = -\frac{2r}{2q - \frac{3}{4}\rho A^2}. \qquad (3.10)$$

By substituting the values of (3.9) and (3.10) in (3.8), we get

$$\left(\frac{dR(A)}{dA}\right)_{R(A)=0} = \frac{3\rho A\left[\omega^2(4r^2\tau + 2r) + 2r\omega_0^2 + \tau(\omega^2 - \omega_0^2)^2\right]}{2\left(2q - \frac{3}{4}\rho A^2\right)\left[\omega^2(2r\tau + 1) + \omega_0^2\right]}. \qquad (3.11)$$

The point A = 0 is a bifurcation point, since

$$\left(\frac{dR(A)}{dA}\right)_{A\to 0} = 0. \qquad (3.12)$$

The oscillation is stable in the small, since the quantity $dR(A)/dA$ remains positive as $A \to 0$. According to Bautin [11], the boundary is "safe" in the small. The point $A \to \infty$ is also a bifurcation point since

$$\left(\frac{dR(A)}{dA}\right)_{A\to\infty} = -0. \qquad (3.13)$$

Thus the question of the stability or instability of the oscillations for $A \to \infty$ remains an open one, and requires a more detailed consideration, i.e., investigation of higher approximations and taking into account of oscillations of the form $A_i \cos \omega_i t$ ($i = 2,3,\ldots,n$). This investigation does not lie within the scope of the present paper.

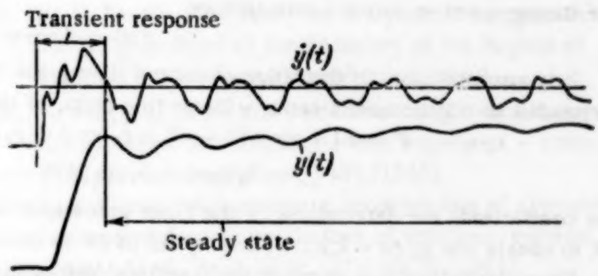

Fig. 4. Oscillogram of the steady-state periodic process in a nonlinear system with lags.

In conclusion, we compare our results with those obtained by Minorsky [6-8]. This latter author considered the appearance of self-excited oscillations in dynamic systems which were described both by linear and by nonlinear differential equations with lagged arguments of the form of (2.7), i.e., essentially linear and nonlinear problems. In considering linear problems, the author arrived at the conclusion that there exist an infinite number of frequencies of self-excited oscillations. In practice, however, in his words, only the basic frequency of the self-excited oscillations is ordinarily observed, which he also considers in his investigation of nonlinear problems. It was shown in the present work that in an autonomous dynamic system described by (2.7) there may exist, not only the fundamental self-excitation frequency, but higher frequencies as well, although their number is strictly finite. The periodic solutions are obtained in the form of sums of series of small parameters, and an estimate of the degree of approximation can always be obtained from the remainder term. Use of the Krylov-Bogolyubov method, in conjunction with the Routh-Hurwitz criterion, allows one to consider, moreover, the boundary of the region of stability of (2.7) and to determine its "dangerous" and "safe" segments. It should be mentioned that the theoretical results obtained from the investigation of the Minorsky equation by means of the Krylov-Bogolyubov asymptotic method were confirmed by simulating the system under consideration on an electronic analog computer. Figure 4 shows the oscillogram of the steady-state periodic process (autooscillation) arising on the boundary of stability of the system considered.

The author wishes to express his deep gratitude to A. M. Letov and Ya. Z. Tsypkin for a number of valuable remarks which have been incorporated in the present work.

APPENDIX

Equivalent Linearization of a Nonlinear Element with Lags by the Harmonic Balance Method

Let the function of the nonlinear element to be linearized have the form

$$F = \varphi\,[\sigma(t-\tau)]. \tag{A.1}$$

while its argument σ varies harmonically:

$$\sigma(t) = A\cos\omega t = A\cos\psi. \tag{A.2}$$

It is assumed that, in the range of control $2|\bar{\sigma}|$, the function F satisfies the Dirichlet conditions and may be expanded in trigonometric series without free term, of the form

$$F = \varphi\,[A\cos\omega(t-\tau)] = \bar{g}_1\cos\psi + \bar{g}_2\cos 2\psi + \cdots + \bar{g}_n\cos n\psi + \cdots \\ \cdots + \bar{h}_1\sin\psi + \bar{h}_2\sin 2\psi + \cdots + \bar{h}_n\sin n\psi + \cdots, \tag{A.3}$$

whose coefficients are determined by the Euler process, in which both members of (A.3) are multiplied by cos $n\psi d\psi$ to obtain the $\bar{g}_n$ ($n = 1,2,\ldots$), and by sin $n\psi d\psi$ to obtain the $\bar{h}_n$ ($n = 1,2,\ldots$), and then integrated from 0 to $+2\pi$. Then, by taking into account the functions' orthogonality properties, we get

$$\bar{g}_n = \frac{1}{\pi}\int_0^{2\pi}\varphi\,[A\cos\omega(t-\tau)]\cos n\psi\,d\psi. \tag{A.4}$$

$$\bar{h}_n = \frac{1}{\pi}\int_0^{2\pi}\varphi\,[A\cos\omega(t-\tau)]\sin n\psi\,d\psi. \tag{A.5}$$

We give sin $n\psi$ and cos $n\psi$ in the form

By substituting the expressions thus obtained in (A.4) and (A.5), we get

$$\bar{g}_n = \frac{\cos n\omega\tau}{\pi}\int_0^{2\pi}\varphi\,[A\cos\psi]\cos n\,d\psi - \frac{\sin n\omega\tau}{\pi}\int_0^{2\pi}\varphi\,[A\cos\psi]\sin n\psi\,d\psi, \tag{A.6}$$

$$\bar{h}_n = \frac{\cos n\omega\tau}{\pi}\int_0^{2\pi}\varphi\,[A\cos\psi]\sin n\psi\,d\psi + \frac{\sin n\omega\tau}{\pi}\int_0^{2\pi}\varphi\,[A\cos\psi]\cos n\psi\,d\psi \tag{A.7}$$

or

$$\bar{g}_n = g_n\cos n\omega\tau - h_n\sin n\omega\tau. \tag{A.8}$$

$$\bar{h}_n = h_n\cos n\omega\tau + g_n\sin n\omega\tau. \tag{A.9}$$

where g_n and h_n are the ordinary coefficients of the Fourier series for the nonlinear function without lags. We now substitute (A.8) and (A.9) in (A.3) and group terms with identical Fourier series coefficients. By using the theorem concerning sines and cosines of a doubled angle, we obtain

$$F = \varphi\,[A\cos\omega(t-\tau)] = g_1\cos\omega(t-\tau) + g_2\cos 2\omega(t-\tau) + \cdots + h_1\sin\omega(t-\tau) + \\ + h_2\sin 2\omega(t-\tau) + \cdots \tag{A.10}$$

In the sequel we shall use either the form in (A.3) or the form in (A.10). Let the nonlinear element be given in the form of (1.2) and the equation of the equivalent linear element be written in the form of (1.3). We use the harmonic balance principle to determine $a_1(A)$ and $a_2(A)$. By substituting (A.2) in (1.2) and taking into account only the fundamental (first) harmonic of the oscillation, we get

$$f\,[A\cos\omega(t-\tau)] = kA\cos\omega(t-\tau) + \varepsilon\bar{g}_1\cos\psi + \varepsilon\bar{h}_1\sin\psi = \\ = Ak(\cos\psi\sin\omega\tau + \sin\psi\sin\omega\tau) + \varepsilon\bar{g}_1\cos\psi + \varepsilon\bar{h}_1\sin\psi, \tag{A.11}$$

where g_1 and h_1 are the coefficients of the trigonometric series defined for the case n = 1 by (A.8) and (A.9). We now compare the oscillation of (A.11) with the oscillations obtained by substituting (A.2) in (1.3). From a comparison of the coefficients of the sines and cosines we obtain two equations for determining the unknown coefficients a_1 and a_2:

$$a_1(A)\,A = Ak\cos\omega\tau + \varepsilon\bar{g}_1, \quad -a_2(A)\,A\omega = Ak\sin\omega\tau + \varepsilon\bar{h}_1. \tag{A.12}$$

From whence

$$a_1(A) = \left(k + \frac{\varepsilon g_1}{A}\right)\cos\omega\tau - \frac{\varepsilon h_1}{A} - \sin\omega\tau, \tag{A.13}$$

$$a_2(A) = -\left(k + \frac{\varepsilon h_1}{A}\right)\frac{\sin\omega\tau}{\omega} - \frac{\varepsilon h_1}{A}\cos\omega\tau, \tag{A.14}$$

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ON STEADY-STATE MODES IN AUTOMATIC CONTROL SYSTEMS*

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The present paper discusses certain basic concepts and methods of automatic control system (ACS) theory which are related to the existence of steady-state modes of autonomous systems with the coexistence of several independent frequencies.

We limit ourselves to the consideration of the broadest class of autonomous ACS, the properties of which are described by a system of ordinary differential equations, given in the Cauchy form:

$$x = f_k(x_1, x_2, \ldots, x_n) \qquad (k = 1, 2, \ldots, n), \quad (1)$$

where the right member lies in some previously given class of functions, defined in a fixed region G, and does not explicitly contain the time parameter t.

Such systems were called dynamic by Birkhoff [1].

Lyapunov [2] called each (undisturbed) solution of (1) (following Routh's example) a steady-state solution with respect to the initiation of disturbance†. Subsequently, however, it turned out that such a general term was inconvenient, and it fell into desuetude. It developed that it was more convenient to apply the term "steady-state modes" to the more specialized motions of (1) which correspond to stable equilibrium positions (or points of rest). In the investigation of an ACS a basic role is played by the problem of determining the steady-state modes (for example, resting points, periodic, almost periodic, and the most general recurring [1] motions) which are stable with respect to other, transient, motions which approximate to these steady-state motions (control quality). All other motions (for example, generation or autooscillation of missile steering, generation of incorrect alignment of the autonomous contour of a power drive or steering mechanism) turn out to be undesirable, and they are considered from the point of view of the necessity for finding the limitation to be imposed on the ACS which will prevent their appearance in the operating modes (from whence the interest in the tuning of the controller's transfer ratio, the determination of the possible external noise conditions, etc.).

The introduction of the concept of the steady-state mode was an obvious step for linear or for two-dimensional nonlinear ACS with continuous right members of (1), but this was definitely not so for two-dimensional ACS with discontinuous right members, and even less so for multidimensional nonlinear ACS. For these, other fundamental concepts also seemed unclear, concepts such as establishment (of the steady-state mode), roughness (of adjustment), physical realizability, and others.

It was known even in A. Poincaré's time that the two-dimensional system (1) with continuous right members could have, in the phase plane (in the given case, in the velocity plane of A. Liénard), only two forms of steady-state modes: an equilibrium position with one state, or a periodic mode with many states.

If the right members of the two-dimensional system are discontinuous, then oscillatory motions of a new type, not converging to those mentioned above, might appear (for example, such motions appear in the systems treated in [3,4]).

In multidimensional systems described by (1), even with continuous right members, guaranteeing the existence (for t from $-\infty$ to $+\infty$) of a unique and continuous dependence of the solution on the initial conditions, there might appear very complicated oscillatory modes, with the coexistence of several, and even infinitely many, incommensurable frequencies.

No mention was made of the existence of such modes even in such fundamental textbooks as [5-7]. Moreover, Teodorchik [8] asserted‡ that, in many cases, one of the frequencies must, in the course of time, dominate over the others, such that after the lapse of some time the (established) mode will be nonetheless periodic (i.e., in actual systems, almost periodic modes do not exist).

Such circumstances led to an incorrect estimate of the value of studying the aforementioned modes in designing ACS. In particular, the problem of uncovering the very simplest of them – the almost periodic ones – in an ACS in the case $n \geq 3$ was completely undeveloped, although the properties of such modes in differential equations were described by such authors as Poincaré, d'Anjou, Hayashi, Nemytskii [9], and others.

The etiology of the longstanding incorrect opinion as to the nonexistence of actual steady-state modes with

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†Lyapunov assumed that the f_k were analytic functions.

‡However, his proofs suffered from mathematical incorrectness.

many states of a more complicated nature than periodic ones is related to a very essential factor having to do with experimental observations, as well as to the widespread use of the harmonic balance method without account being taken of its domain of permissible applicability. The fact of the matter is that, in making our observations, we cannot distinguish between, for example, periodic and almost periodic steady-state modes, since the accuracy of the instruments and the intervals of observation time are quite limited. These individually noncharacterizeable steady-state modes present themselves to us phenomenologically as recurrent motions, i.e., those which were introduced into consideration as the most general motions by Birkhoff [1] in 1928. They are characterized by the property that, for each $\epsilon > 0$, there exists its own interval of observation T such that any trajectory of duration T of the recurrent steady-state mode will contain in its ϵ-envelope all the trajectories (as time varies from $-\infty$ to $+\infty$). Thus, an operator taking an observation during a time interval greatly exceeding T will always touch upon a "diffuse cycle" (a periodic mode with noise).

However, from the heuristic point of view, it is not at all a matter of indifference whether we have to do with a periodic steady-state mode, distorted by random noise or, for example, with an almost periodic mode. In the first case, an analytic investigation of the ACS is possible; in the second case, such an investigation is frequently impossible for the reasons previously cited.

Not only Birkhoff but, in the 1930's, Andronov **, spoke of the importance of investigating almost periodic steady-state modes. Finally, Aizerman [10] came out convincingly in defense of the physical reality of these steady-state modes and the necessity of studying them analytically for designing ACS.

1. Definition and Internal Properties of Dynamic Systems

We preface the more graphical explication and illustrations by a somewhat abstract definition of the dynamic system given by (1) and by a description of certain of its properties, borrowed from topological dynamics [11]. This digression allows an exact mathematic formulation to be given, but does not impair the understanding of the following simpler material.

Let X be a topological space of certain elements $x \in X$, which we shall call a phase space, and let the $\underline{x}$ be phases (points). Further, let T be a topological group of elements $t \in T$ (such that every pair $t_1, t_2 \in T$ is put into correspondence with a third element $t_3 \in T$). We call T the time group, with moments $\underline{t}$. In X we define a transformation F which puts each point $x \in X$ into correspondence with another point of X which we denote by $xF \in X$. The topological product XT is mirrored by the transformation F in X, i.e., each pair (x,t), where $x \in X$ and $t \in T$, corresponds to a point (x,t)F. Descriptively (axiomatically), we define the topological transformation group, denoted briefly by the triplet (X,T,F), which satisfies the following axioms:

1. Identity axiom. In the group T there exists an initial moment of time $0 \in T$ (0 need not be a number) such that the pair $\underline{t}$ and 0 corresponds to $\underline{t}$, and the pair (x,0) is put into correspondence, by transformation F, with the point (x,0)F = x.

2. Homomorphism axiom. If the pair t_1, $t_2 \in T$ corresponds to $t_3 \in T$, then $[(x, T_1)F, t_2]F = (x, t_3)F$.

3. Continuity axiom. The transformation F is one-to-one and continuous (i.e., topological).

If X is a metric space with metric ρ, and the moments of time $t \in T$ are real numbers then, in accordance with the continuity axiom, (x,t)F is a continuous function of the collection (x,t), i.e., for each $t \in T$ and a given pair of numbers $\epsilon > 0$ and $x \in X$ there exists another pair of numbers $\delta = \delta(x,t) > 0$ and $\tau = \tau(x,t)$ such that, for any $y \in X$ and $t_1 \in T$ such that $\rho(x,y) < \delta$ and $|t-t_1| < \tau$, the following inequality holds: $\rho[(x,t)F, (y,t_1)F] < \epsilon$.

By a trajectory (motion) we mean the geometric locus of the images of the fixed point $x \in X$ under the transformation F, corresponding to all the $t \in T$.

If, for all $t \in T$ the transformation F takes the set $I \subset X$ into itself, then I is called an invariant. For example, a singular point (a rest or equilibrium state), each trajectory, the set of all trajectories passing through all the limit points of a fixed almost periodic trajectory (in the case of a metric space X, cf. below), are invariant sets.

It is easily seen that the triplet (I,T,F) satisfies the three axioms given above and, consequently, is itself a topological transformation group (with the induced topology of X).

A nonempty closed invariant set I is called minimal (I_{min}) if it does not contain an invariant set as a proper subset. In the example given above, the invariant set of almost periodic trajectories is minimal.

If, on I_{min}, the invariant J is defined with respect to the points $t \in T$ then we say that I_{min} possesses J-internal stability.

Each trajectory (or resting point) possesses a natural (trivial) invariant: the transformation F takes it into itself for all $t \in T$.

Turning now to the other side, we consider metric invariants. For this we assume that X is a metric space with the metric ρ, and the moments of time $t \in T$ are real numbers. The set I_{min} has S-internal stability (in the sense of Franklin [12]) if, for any pair $\epsilon > 0$ and $x \in I_{min}$ there exists $\delta = \delta(\epsilon, x) > 0$ such that, for each $y \in I_{min}$ lying in the δ-neighborhood of $\underline{x}$ [i.e., for $\rho(x,y) < \delta$], there will hold, for all moments of time $t \in T$ (i.e., for $-\infty < t < +\infty$

$$\rho[(\dot{x}, t)F, (y, t)F] \leqslant \varepsilon. \qquad (2)$$

If, moreover, I_{min} is bounded, i.e., there exists a phase $y \in X$ such that, for each $x \in I_{min}$, $\rho(x, y) < \text{const} < +\infty$, then, by means of the Vitali covering theorem, it is easily proven that in this case δ will depend only on ϵ,

**This latter was communicated to the author by El'shin who consulted with Andronov during those years.

and I_{min} will represent a recurrent motion (in the sense of G. D. Birkhoff [1]), i.e., for each pair $\varepsilon > 0$ and $x \in I_{min}$ there will exist a $\tau_* = \tau_*(\varepsilon, x) > 0$ such that, for any other pair $t_0, t \in T$ one finds a number $\tau = \tau(t) \in (t_0 - \tau_*, t_0 + \tau_*)$. for which

$$\rho[(x, t)F, (x, \tau)F] < \varepsilon. \tag{3}$$

Then, a fortiori, I_{min} will consist of L-internal stable trajectories (in the sense of Lagrange), i.e., there will exist a $y \in X$ such that, for each fixed $x \in I_{min}$ and all $t \in T$

$$\rho[(x, t)F, y] < \text{const} < +\infty. \tag{4}$$

Each recurrent trajectory is P-internally stable, meaning that it possesses the local recurrence property (repeatability) in contradistinction to the integral property of (3) (in the sense of Poisson [1]). i.e., for any pair $\epsilon, \tau > 0$ and any $x \in I_{min}$ there exist two moments of time, $t_1 < -\tau$ and $t_2 > \tau$ for which

$$\rho[(x, t)F, x] < \varepsilon. \tag{5}$$

The converse is not true. It does not follow from the P-internal stability given by (5) that the trajectory is recurrent, as given by (3).

Since a continuous function on a closed set is uniformly continuous, it follows from the continuity axiom and Vitali's covering theorem, that a closed trajectory (cycle) has S-internal stability and, consequently, presents a recurrent motion.

Markov [13] showed that each bounded S-internally stable I_{min} consists of almost periodic trajectories, i.e., trajectories such that each of their components is a Bohr almost periodic function, i.e., is expanded in a trigonometric series

$$\sum_{i=1}^{\infty} (a_i \sin \omega_i t + b_i \cos \omega_i t), \tag{6}$$

where the a_i and b_i are constants, and the numbers ω_i need not be integers (in contradistinction to Fourier series).

The following property holds for an almost periodic trajectory $(x,t)F$. For any $\epsilon > 0$ there exists a $\tau_* = \tau_* \cdot (\epsilon) > 0$ such that, for each $t_0 \in T$ in the semi-interval $[t_0, t_0 + \tau_*]$ there exists at least one number τ (the so-called almost-period with respect to ϵ) for which

$$\rho[(x, t+\tau)F, (x, t)F] \leqslant \varepsilon \tag{7}$$

for all $t \in T$.

In particular, if (7) holds for $\epsilon = 0$, then the trajectory is called periodic (a cycle) with period τ. In this case, all the ω_i ($i = 1,2,\ldots$) in (6) will be integers.

We note that dynamic ACS (1) may not satisfy the three enumerated axioms. For example, let X be the ordinary n-dimensional Euclidean space, T the set of real numbers, and $\varphi(x)$ a continuous decreasing function such that $\varphi(x_0) < 0$ for some $x_0 > 0$. Then, the trajectory defined by the equation $\dot{x} = \varphi(x)$ does not exist (i.e., can not be continued) for $t < \int_{x_0}^{+\infty} dx/\varphi(x) < 0$, if this latter integral exists.

It is known that each L-internally stable trajectory (1) can be continued.

A trajectory of the type in (1) that can be continued and is uniformly continuous in t is S-internally stable since, in this case, for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that (7) holds for $0 < \tau < \delta$ and any t.

Local continuity in t is a necessary but not a sufficient condition for S-internal stability. For example, the trajectory $y = 1/(1-x)$ of the system $\dot{x} = 1$, $\dot{y} = -1/(1-x)^2$ which passes through the point (0,1) for t = 0 does not possess S-internal stability.

We mention still one more important circumstance. A trajectory of type (1) which can not be continued is necessarily unbounded. The converse is not true. There exist trajectories which are simultaneously S-internally stable and L-internally unstable. Tricomi [14] first encountered such motions in a rotor (armature) of a synchronous motor which lagged the stator's rotating magnetic field by the angle θ, the motion being described by the equation

$$\ddot{\theta} + A\dot{\theta} + B\sin\theta = C. \tag{8}$$

where $A > 0$, $B > C > 0$. The law of this motion is defined by the continuous periodic function $\vartheta = \vartheta(t)$ with period τ:

$$\theta = \vartheta + \frac{2\pi}{\tau} t. \tag{9}$$

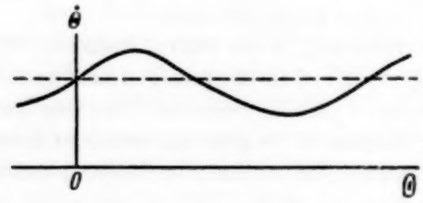

Fig. 1

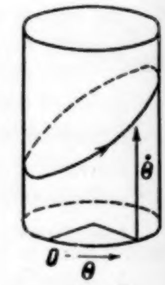

Fig. 2

Fig. 1. Slipping mode of an asynchronous motor's rotor.

Fig. 2. Cylindrical phase plot of an asynchronous motor's rotor slippage.

On the phase (velocity) Liénard $\{\theta, \dot{\theta}\}$ plane, this law takes the form of the unbounded curve shown in Fig. 1.

This law of motion also has another geometric interpretation. On a phase cylinder (Fig. 2), it is given by a closed curve (a cycle) with elliptical form, and is bounded.

The dual interpretation just given has a curious meaning. According to Nemytskii's classification [9], each bounded S-internally stable and (simultaneously) asymptotically stable (external, in the sense of A. M. Lyapunov) trajectory represents a steady-state mode. Consequently, in the sense of Nemytskii, the mode given by (9)††, depending on the choice of the coordinate system, is simultaneously both steady-state and not steady-state.

††It is known that the mode of (9) is asymptotically stable in the sense of Lyapunov.

2. Definition of the Steady-State Mode

We assume *a priori* that, in topological space X, a limiting operation is defined, and that the group T is a well-ordered set of elements ‡‡.

Let I_{min} be embedded in the open region U. The set of points $\underline{x}$ which lie in U but do not lie in I_{min} is called the open envelope of I_{min}, and is denoted by $U(I_{min})$.

If there exists a $U(I_{min})$ such that, for each of its points, there is an increasing sequence $t_i \in T$ $(i = 1, 2, \ldots)$

$$t_1 < t_2 < \ldots < t_n < \ldots \tag{10}$$

such that

$$(x, t_n) F \to I_{min} \text{ for } n \to +\infty, \tag{11}$$

then I_{min} is called orbitally asymptotically stable (in the sense of Poincaré) in the positive direction. In this case, $U(I_{min})$ is called the domain of attraction of I_{min}.

Correspondingly, one speaks of a negative direction, if the sequence in (10) is a decreasing one.

If there exist simultaneously two sequences, one increasing and the other decreasing, one says that I_{min} is simply orbitally asymptotically stable.

If, moreover, there exists a point $y \in I_{min}$ such that, in addition to (11), there holds the relationship

$$(x, t_n) F \to (y, t_n) F \quad \text{for} \quad n \to +\infty, \tag{12}$$

then one says that I_{min} is asymptotically stable in the sense of A. M. Lyapunov [in, respectively, the positive, negative, and in both directions where, in the latter case, one says that I_{min} is unconditionally asymptotically stable (in the sense of G. Dirichlet)].

If the space X is a metric one, then the definitions just given assume their ordinary meanings***.

Each mode of motion which is an internally and orbitally asymptotically stable (externally) I_{min} is called a steady-state mode. The corresponding process of establishing the steady state is defined by the limiting process of (11) in the sense of the topology of X.

Each I_{min} of steady-state modes has a domain of attraction. For example, such domains do not exist for any invariant set of motions defined by the equation $\ddot{x} + x = 0$.

Nemytskii [9] showed that the presence of a domain of attraction in an almost periodic mode essentially simplifies its structure. Thus, for example, each asymptotically stable (in the sense of Lyapunov) steady-state mode of a three-dimensional system is either an equilibrium position (with one state) or a periodic mode or an almost periodic mode (with many states) having not more than two independent frequencies. The images of these motions in phase space are, respectively, a point, a closed curve, and an irrational toroidal winding. Whether the analogous suppositions hold for n-dimension systems, for $n > 3$, is not yet known, but is probable.

The domain of attraction of an almost periodic steady-state mode can not keep its trajectories isolated. An infinite set of trajectories passing through the limit points of these almost periodic trajectories will also be embedded in the domain of attraction. All these trajectories will also be almost periodic, with the same frequencies as the first one. The trajectory of a periodic steady-state mode can be unique in its domain of attraction.

We note in conclusion that Nemytskii [9], following Franklin [12], insists that each steady-state mode must be S-internally stable. He starts from the condition that "if at some moment of time two states were neighboring, it must follow that in the future these states must remain neighboring, since otherwise the mode under investigation either would have in the past, or will have in the future, some tendency to development consisting of a separation of these two states from each other, which would render it impossible to speak of a steady-state mode". These words seem essentially unconvincing and unclear, so that the development in time must be understood only from the point of view of the limitation given in (2).

Since every motion in time is a development, then each temporal invariant may, with equal justification, be taken as an indication of nondevelopment (in the meaning defined by him), stationary nature, or establishment in time. The choice of some temporal invariant or another must be determined, not by subjective conceptions, but by the objective problem which supervenes in the concrete investigation of an ACS. On this score, one could give many convincing examples from practice.

3. On Certain Methods of Investigating Almost Periodic Steady-State Modes in an ACS

The simplest method of investigation is a modification of Minorsky's approximate "stroboscopic" method [16], well-known in nonlinear mechanics, this method being a further development of the principle introduced into science in 1883 by A. Gulden and Linstedt [17]†††. The limits of applicability of the "stroboscopic" method were determined by Urabe [18].

For greater clarity we demonstrate this method by the example of a four-dimensional ACS without, however, lessening the generality.

We turn to the example of a "blind" winged missile, circling (without lateral banking) at a constant altitude under the action of residual voltage U (of a hysteresis magnetic or "zero drift" vacuum tube amplifier).

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‡‡The topological transformation group (X,T,F) is a semi-ordered set if T is well-ordered [11].

***We note that, in the monograph of Zubov [15], orbital stability, as given by (11), is called stability in the sense of Lyapunov, which can lead to misunderstandings. For example, for an isolated orbit of a heavenly body (a Sputnik moving in accordance with Kepler's law in a conservative central force field $\ddot{r} = -\mu r/|r|^3$ (where $\underline{r}$ is the vector of the body and μ is the gravitational constant), the orbit is unstable in the sense of Lyapunov and orbitally stable.

†††Their work also served as the starting point for the development of the results of Krylov and Bogolyubov.

In the particular case, the missile equations can be put in the form

$$\ddot{\beta} + c_1\dot{\beta} + c_2\beta + c_3\delta = 0. \qquad (13)$$
$$\dot{\psi} = \dot{\beta} + c_4\beta$$

(where c_1, c_2, c_3, and c_4 are positive numbers, and β, ψ and δ are the angles of lateral slipping, course, and rudder deviation from the neutral position), while the autopilot equations are

$$\dot{\delta} = f(\sigma), \quad \sigma = i(\dot{\beta} + \nu\beta - U) - \delta. \qquad (14)$$

where $f(\sigma)$ is the piecewise-smooth velocity characteristic of the steering mechanism [$\sigma f(\sigma) > 0$ for $\sigma \neq 0$, $f(0) = 0$], $i > 0$ is the autopilot's rigid feedback coefficient and $\nu > 0$ is the transfer ratio of the accelerometer for lateral linear acceleration.

We assume a priori that oscillations are observed in the system, and that the conditions for applying the "stroboscopic" method are satisfied ‡‡‡. We may then set

$$\beta = \beta_0 + x(t)\sin nt + y(t)\cos nt, \quad \sigma = z(t)\sin nt, \qquad (15)$$

where $\beta_0 = c_3 U/(c_3\nu + c_2/i)$. n is angular frequency, and $x(t)$, $y(t)$, and $z(t)$ are slowly varying functions of time (in relation to the period $2\pi/n$).

On the basis of the previously given theorem of V. V. Nemytskii [9], the state equilibrium positions and the periodic modes of the $\{x, y, z\}$ phare space of the stroboscopic projections of the hyperplanes of the $\{\beta, \dot{\beta}, \delta\}$ phase space of (13)-(14) completely define all the bounded steady-state modes (possessing S-internal stability) of (13)-(14) [owing to the degeneration of (13), since the second equation does not depend on the first]. A study of the three-dimensional (induced) system of equations in the $\{x, y, z\}$ phase space can be carried out by the methods presented in [19-25] and others. However, there are as yet no effective methods for determining the domains of attraction (except for what was presented in [23]).

It so happens that, under certain conditions, the $\{x, y, z\}$ phase space contains a unique stable cycle. It is then easily seen that the cylindrical $\{\beta, \dot{\beta}, \delta, \psi\}$ phase space of (13)-(14) will contain a unique (bounded) steady-state almost periodic mode with two coexisting independent frequencies n and $c_4\beta_0$, so long as the latter are incommensurable.[1]

This same result is attained by the (exact) method of the topological product, the essence of which is as follows.

By well-known topological methods [23] one proves the existence of a stable oscillatory mode in $\{\beta, \dot{\beta}, \delta\}$ phase space.

The cylindrical $\{\beta, \dot{\beta}, \delta, \psi\}$ phase space is considered as the topological product of the $\{\beta, \dot{\beta}, \delta\}$ hyperplane and the $\{\psi\}$ circle. By this we arrive at the result given above

It should be mentioned that the topological product method has thus far been applied only to those ACS whose equations are either degenerate, or reduce in some fashion to those groups of which one, at least, is amenable to the application of one of the methods of investigation given in [19-25].

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‡‡‡For example, in the particular case when $f(\sigma) \equiv A\sigma + B\sigma^3$, where the constants $A > 0$ and $|B|$ are sufficiently small.

[1]If the $(\beta, \dot{\beta}, \delta, \psi)$ space is considered as being Euclidean, then an almost periodic mode turns out to be unbounded, i.e., it will not be steady-state in the sense of Nemytskii (as in the example of Tricomi given earlier).

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[2]See translation.

DETERMINING PULSE SYSTEM OPTIMUM WEIGHTING FUNCTION WHICH ENSURES THE EXTREME OF A FUNCTIONAL

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A method is described for finding the optimum pulse system which ensures an extreme of a sufficiently general criterion in the form of a functional $I = \Phi(I_1, \ldots, I_{n+1})$, where Φ is a known function, and I_i are quadratic functionals of the weighting function of the system.

The methods for the determination of the optimum linear pulse systems minimizing the mean-square-error at its output are given in [1,2]. The least mean-square-error criterion seems a suitable choice in many cases when selecting a dynamic system. In practice, however, when evaluating and comparing various pulse systems one often encounters cases where the most suitable criterion is a more complex probability characteristic of the accuracy of the system operation. A characteristic of this type may be, for example, the probability P of the system error not exceeding by numerical value a certain given quantity $\underline{c}$. In a specific case the probability P may usually be expressed in terms of mathematical expectation and variance of the separate components of the error, the components having different distributions. (When the components of the system error also depend on one another the moments due to their correlation should also be considered. For example, if the error of the system is composed of the sum of mathematical expectation $\underline{a}$ and a random component normally distributed with variance σ^2, then

$$P = \frac{1}{\sqrt{2\pi}\sigma} \int_{-c}^{c} \exp\left[-\frac{(a+x)^2}{2\sigma^2}\right] dx = I(a^2, \sigma^2). \tag{1}$$

In the case when the mathematical expectation of the error is $\underline{a}$, and the first random component Z_1 of the error is distributed normally with variance σ^2, and the second random component Z_2 has a rectangular distribution within the interval from $-b$ to $+b$, and Z_1 and Z_2 are independent, then

$$P = \frac{1}{\sqrt{2\pi}2b\sigma} \int_{-c}^{c} \left\{ \int_{-b}^{b} \exp\left[-\frac{(a+y+x)^2}{2\sigma^2}\right] dx \right\} dy = I(a^2, \sigma^2, b^2). \tag{2}$$

In the often encountered case when the quantity $\underline{c}$ is small, (1) and (2) assume a simpler form:

$$P \approx \frac{\sqrt{2}c}{\sqrt{\pi}\sigma} \exp\left[-\frac{a^2}{2\sigma^2}\right], \tag{1a}$$

$$P \approx \frac{c}{\sqrt{2\pi}\sigma b} \int_{-b}^{b} \exp\left[-\frac{(a+y)^2}{2\sigma^2}\right] dy. \tag{2a}$$

The quantities a^2, σ^2, b^2 are quadratic [3,4,6] functionals of the system weighting function. In a more general case a^2, σ^2, b^2 also depend parametrically on time. Consequently, even the optimum system which at any given moment of time is extreme depends on time and is thus nonstationary.

We shall assume henceforth that the criterion I for selecting the optimum system is a given function Φ of n+1 quadratic functionals I_i ($i = 1, \ldots, n+1$) which in turn depend on the weighting function of the pulse system. The problem of determining the optimum linear pulse system according to the I criterion is formulated in the same way as in [3,5] when determining the optimum linear system of a continuous type.

We assume that there is a method of finding an optimum weighting function $k_0[n, m, \epsilon]$* of a pulse system with given time of the transient process and such that the function will ensure the extreme of the quadratic functional

$$I_{II} = \theta_1 I_1(k[n, m, \varepsilon]) + \ldots + \theta_n I_n(k[n, m, \varepsilon]) + I_{n+1}(k[n, m, \varepsilon]) \tag{3}$$

for arbitrary values of the parameters θ_i. A method is sought to determine the optimum weighting function $k^\circ[n, m, \epsilon]$ with the same transient process time interval, and such that it will ensure the extreme of the functional

$$I = I(k[n, m, \varepsilon]) = \Phi(I_1, \ldots, I_{n+1}). \tag{4}$$

The time is here expressed in integral units [1,2]. The time in conventional units is obtained from the integral units as follows: $t = nT + mT + \epsilon T$ at any moment of time $t = mT$ at the discrete values of time. The quantity T is the pulse sampling period, otherwise the control time interval of the pulse system, and ϵ varies from 0 to 1.

*The weighting function $k[n, m, \epsilon]$ of a pulse system with variable parameters of the equivalent (reducible) continuous part expresses its response at the moment $t = n + m + \epsilon$ to a unit impulse applied to the input at the moment $t = n - 0$.

The problem is solved at an arbitrary moment of time regarded as a parameter.

The difference between the formulations of this problem and of [3,5] lies in the latter continuous case being the optimum system in the class of all the possible linear systems with the same given duration of the transient process;and in our problem the optimum system is selected from the class of pulse linear systems with given pulse sampling period as well as duration of transient process. All the pulse system weighting functions with a given pulse sampling period form a linear manifold (which also holds good for weighting functions of continuous linear systems)†. Therefore, all the derivations and proofs obtained in [3,5] for linear systems of continuous type could be repeated by replacing everywhere "linear system" (of continuous type) with "pulse system" (with given pulse sampling period). We just state the fundamental result without reproducing the derivations.

The necessary conditions which must be fulfilled by the weighting function $k^\circ[n,m,\epsilon]$ ensuring the extreme of the functional I are:

a) The function k° should ensure the extreme of the functional I_{II}, i.e., it is required that:

$$k^\circ = k_0[n,\ m,\ \varepsilon,\ \theta_1,\ \ldots,\ \theta_n]\,|_{\theta_i=\theta_{i0}}; \tag{5}$$

b) The following should be valid:

$$\frac{\partial I\,(k_0\,[n,\ m,\ \varepsilon,\ \theta_1,\ \ldots\ \theta_n]}{\partial\theta_i}\Bigg|_{\theta_j=\theta_{j0}} \tag{6}$$

$$(i,\ j=1,\ \ldots,\ n).$$

or (a special case) the condition below must be fulfilled:

$$\frac{\partial\Phi\,[I_1\,(k^\circ),\ \ldots\ I_{n+1}\,(k^\circ)]}{\partial I_l}=0 \qquad (l=1,\ \ldots\ n+1).$$

which illustrates the fact that the function Φ reaches its extreme within the domain of variability of the I_i functionals. This special case is usually of no practical importance.

The condition a) enables one to reduce the complex task of determining the function k° at which the extreme of I occurs to a less complex one of finding k_0 at which the extreme of I_{II} occurs. Determining the function $k_0 = k_0[n,m,\epsilon,\theta_1,\ldots,\theta_n]$ is the principal stage in the solution of our problem. Once the function $k_0[n,m,\epsilon,\theta_1,\ldots,\theta_n]$ has been obtained explicitly for arbitrary values of the parameters θ_i, it is easy to determine the values of the parameters $\theta_i = \theta_{i0}$ satisfying the condition b)‡. Where only approximations to the function are obtained, the values of the parameters $\theta_i = \theta_{i0}$ are also evaluated approximately, for example, graphically. As long as the number $\underline{n}$ of the parameters θ_i remains small, the determination of the values θ_{i0} is fairly straightforward and easy. This is why the determination of the weighting function $k_0[n,m,\epsilon,\theta_1,\ldots,\theta_n]$ ensuring the extreme of I_{II} should have primary attention. Methods of determining k_0 with data given in other forms have been sufficiently developed [6, 7 and others]. The functionals I, I_i, I_{II} obtained when investigating the pulse systems differ from the corresponding functionals I, I_i, I_{II} obtained when examining the continuous linear systems (the integrals in the latter case correspond to sums in the former one).

In the case of a pulse system, the determination of k_0 can be reduced to the solution of linear algebraic equations [1,2]. As is known, in the continuous case k_0 is determined as a solution of a certain linear integral equation. The actual form of the equations depends in both cases on the way in which the data are given.

We shall consider one method of determining the optimum weighting function of a pulse system from the criteria (1a). Let on the input of the pulse system be impressed a useful signal $X[n,\epsilon]$ which can be represented as a sum of a known nonrandom continuous function $x[n,\epsilon]$ and a random function $X^\circ[n,\epsilon]$ distributed normally with xero mean and correlation function $R_x[n-m, n-l]$. The interference $Z[n,\epsilon]$ also occurs at the input; it is a random function normally distributed with zero mean and correlation function $R_z[n-m, n-l]$. In order to simplify the subsuquent exposition, it is assumed that interference and signal are not correlated, that is, $M[XZ] = 0$. The pulse system at the output is given by the function $HX[n,\epsilon]$, where H is a linear operator. The time of the system transient process is equal to N.

In this case the error is equal to

$$E\,[n,\ \varepsilon]=\left\{\sum_{m=0}^{N}x\,[n-m]\,k\,[n-m,\ m,\ \varepsilon]-Hx\,[n,\ \varepsilon]\right\}+ \tag{7}$$

$$+\left\{\sum_{m=0}^{N}X^\circ\,[n-m]\,k\,[n-m,\ m,\ \varepsilon]-HX^\circ\,[n,\ \varepsilon]\right\}+\sum_{m=0}^{N}Z\,[n-m]\,k\,[n-m,\ m,\ \varepsilon]$$

(the effect of initial conditions being neglected).

In the expression (7) the first term represents the mathematical expectation $\underline{a}$ of the error, the second and the third term being random components of the error.

† A set of M is called linear manifold when its elements satisfy the following rules: when k_1 and k_2 belong to M, then $a_1k_1 + a_2k_2$ also belongs to M, whatever the arbitrary numbers a_1, a_2.

‡ It was shown in [3, 5] that in order to determine $\theta_i = \theta_{i0}$ one can utilize another condition:

$$\frac{\partial I}{\partial I_j}\bigg/\frac{\partial I}{\partial I_{n+1}}=\theta_j \quad (j=1,\ \ldots,\ n).$$

instead of condition b).

We denote by I_1 the variance of the error:

$$I_1 = \sigma^2 = M[(E-a)^2] = \sum_{l=0}^{N}\sum_{m=0}^{N} R_z[n-m, n-l]\,k[n-m, m, \varepsilon]\,k[n-l, l, \varepsilon] +$$
$$+ \sum_{l=0}^{N}\sum_{m=0}^{N} R_x[n-m, n-l]\,k[n-m, m, \varepsilon]\,k[n-l, l, \varepsilon] - \tag{8}$$
$$-2\sum_{l=0}^{N} HR_x[n-l, n+\varepsilon]\,k[n-l, l, \varepsilon] + HHR_x[n+\varepsilon, n+\varepsilon].$$

In the last but one term, the operator H on the correlation function R_x operated with respect to its second argument.

We denote by I_2 the square of the mathematical expectation of the error E:

$$I_2 = a^2 = \{M[E]\}^2 = \left\{\sum_{m=0}^{N} x[n-m]\,k[n-m, m, \varepsilon] - Hx[n, \varepsilon]\right\}^2. \tag{9}$$

The functional I_{II} in this case takes the form

$$I_{II} = \theta_1 I_1 + I_2. \tag{10}$$

Bearing in mind what was stated previously, one has to find a weighting function $k_0[n, m, \epsilon, \theta_1]$ which will ensure the extreme of the functional (10). The necessary condition satisfied by such a function k_0 is [6,8] the vanishing of the first variation of the functional I_{II}

$$\left.\frac{dI_{II}\{k[n, m, \varepsilon, \theta_1] + \Delta\varkappa[n, m, \varepsilon]\}}{d\Delta}\right|_{\Delta=0} = 0 \tag{11}$$

for any weighting function κ of a pulse system with pulse sampling period T and time N of the transient process.

We rewrite the expression for the quadratic functional $I_{II}[k + \Delta\kappa]$ in the expanded form

$$I_{II}[k+\Delta\varkappa] = I_{II}[k] + 2\Delta D_1 + \Delta^2 D_2. \tag{12}$$

where

$$D_1 = \sum_{l=0}^{N} \varkappa[n-l, l, \varepsilon]\left\{\sum_{m=0}^{N} k[n-m, m, \varepsilon, \theta_1](\theta_1 R_z[n-m, m-l] +\right.$$
$$+ \theta_1 R_x[n-m, n-l] + x[n-l]\,x[n-m]) - x[n-l]\,Hx[n, \varepsilon] -$$
$$\left. - \theta_1 HR_x[n-l, n+\varepsilon]\right\},$$

$$D_2 = \sum_{l=0}^{N}\sum_{m=0}^{N} \varkappa[n-l, l, \varepsilon]\,\varkappa[n-m, m, \varepsilon](\theta_1 R_z[n-m, n-l] +$$
$$+ \theta_1 R_x[n-l, n-m] + x[n-m]\,x[n-l]).$$

It follows from (11) and (12) that

$$\sum_{l=0}^{N} \varkappa[n-l, l, \varepsilon]\left\{\sum_{m=0}^{N} k[n-m, m, \varepsilon, \theta_1](\theta_1 R_z[n-m, n-l] +\right.$$
$$+ \theta_1 R_x[n-m, n-l] + x[n-l]\,x[n-m]) - \theta_1 HR_x[n-l, n+\varepsilon] - \tag{13}$$
$$\left. - x[n-l]\,Hx[n, \varepsilon]\right\} = 0.$$

In order that (13) be valid for an arbitrary function κ from the admissible class, it is necessary that the relation given below be valid:

$$\sum_{m=0}^{N} k[n-m, m, \varepsilon, \theta_1](\theta_1 R_z[n-m, n-l] + \theta_1 R_x[n-m, n-l] +$$
$$+ x[n-l]\,x[n-m]) - \theta_1 HR_x[n-l, n+\varepsilon] - x[n-l]\,Hx[n, \varepsilon] = 0 \tag{14}$$
$$(l = 0, 1, \ldots, N).$$

To prove it, let us assume that the relation (14) is not valid for certain $l = l_1$, $0 \le l_1 \le N$; the parameters n, ϵ, θ_1 are assumed constant. When $l = l_1$ the left-hand side of (14) is equal to some $S \neq 0$. Knowing that κ is an arbitrary function, we choose κ in the following way:

$$\varkappa[n-l_1, l_1, \varepsilon] = 1, \ \varkappa[n-l, l, \varepsilon] = 0 \qquad (l \neq l_1).$$

But if κ is chosen in this way, (13) is not true. Consequently, our assumption has been proved to be wrong.

The relation (14) is also a sufficient condition for the function $\underline{k}$ to minimize I_{II} when $\theta_1 \ge 0$. It follows, namely, from (12) that, when $\underline{k} = \underline{k}_0$ satisfies (14), then $I_{II}[k_0+\varkappa] = I_{II}[k_0] + D_2$, where $D_2 \ge 0$.

It follows that $I_{II}[k_0+\varkappa] \geqslant I_{II}[k_0]$ for an arbitrary κ, which is what we had to prove.

The values $\theta_1 < 0$ are, as a rule, of no practical importance.

The solution $k_0[n,m,\epsilon,\theta_1]$ of the simultaneous linear algebraic equations (14) must be substituted in (8) and (9) to obtain the relations $a^2 = a^2(\theta_1)$, $\sigma^2 = \sigma^2(\theta_1)$; subsequently, by using (1a), it is easy to obtain P as a function of θ_1. Then the value $\theta_1 = \theta_{10}$ corresponding to the maximum of P is found from $dP(\theta_1)/d\theta_1 = 0$. The optimum weighting function of a pulse system is in our case: $k^* = k_0[n,m,\epsilon,\theta_{10}]$; the greatest value of probability – which points to the optimum system – is $P_0 = P(\theta_{10})$.

The conditions which ensure the extreme of the functional I actually occurring at k^* are not considered in the present paper. For pulse systems these conditions can be formulated analogously to those in [5] for continuous systems.

When dealing with a complex problem the main difficulty lies in solving the simultaneous equations (14) or similar equations. In a number of special cases of a similar character, the simultaneous equations can be solved exactly [1]. In the general case, the solving of such or similar equations can be accomplished by means of various approximate methods. The method of the steepest descent [4,6] appears to be particularly appropriate. We shall also apply it directly to determine the extreme of the quadratic functional I_{II}. (A description is given of the main features of this method when it is utilized to find the minimum of the functional I_{II}.

Let the selection of the first approximation of $k[n-m, m, \epsilon] = k_1[n-m, m, \epsilon]$, based on arbitrary considerations. We find next the direction of the gradient in the normed space of functions $k[n-m, m, \epsilon]$, that is, the direction along which the functional I_{II} decreases most steeply. The norm of a function $\underline{k}$ is defined by

$$\|k[n-m, m, \varepsilon]\| = \sum_{m=0}^{N} k^2[n-m, m, \varepsilon]. \quad (15)$$

The function $\kappa_1[n-m, m, \epsilon]$, which corresponds to the direction of the gradient, should ensure the extreme of the quantity

$$\frac{dI_{II}(k_1[n-m, m, \varepsilon] + \Delta\varkappa_1[n-m, m, \varepsilon])}{d\Delta}\Big|_{\Delta=0} \quad (16)$$

under the condition

$$\sum_{m=0}^{N} \varkappa_1^2[n-m, m, \varepsilon] = 1. \quad (17)$$

It follows that this function should also ensure the unconditional extreme of the expression

$$\frac{dI_{II}(k_1[n-m, m, \varepsilon] + \Delta\varkappa_1[n-m, m, \varepsilon])}{d\Delta}\Big|_{\Delta=0} + \quad (18)$$
$$+ \mu \sum_{m=0}^{N} \varkappa_1^2[n-m, m, \varepsilon],$$

with μ being the Lagrange multiplier whose value can be found from the constraint (17) (in our problem the multiplier does not appear to be real and therefore is not found from the condition (17)). Bearing in mind that I_{II} is a quadratic functional one is able to write down

$$I_{II}(k_1[n-m, m, \varepsilon] + \Delta\varkappa_1[n-m, m, \varepsilon]) = \quad (19)$$
$$= I_{II}[k_1] + 2\Delta A_1 + \Delta^2 A_2,$$

where A_1, A_2 depend only on κ_1 and κ_2 but do not depend on Δ, A_1 being of the form

$$A_1 = \sum_{m=0}^{N} B_1[n-m, m, \varepsilon]\, L\varkappa_1[n-m, m, \varepsilon].$$

The expression $B_1[n-m, m, \epsilon]$ here depends only on k_1, but not on κ_1; L is a linear transformation (on κ_1) operator. It is easy to see that in the case often encountered in practice, when $L\kappa_1 = \kappa_1$, the necessary condition for the extreme of the functional (18) is, with an accuracy of up to multiplier μ, of the form

$$\varkappa_1[n-m, m, \varepsilon] = B_1[n-m, m, \varepsilon]. \quad (20)$$

As the second approximation of the function $\underline{k}$ one takes

$$k_2 = k_1 + \Delta_1\varkappa_1,$$

where κ_1 is given by (20) and Δ_1 is obtained from the condition that when $\Delta = \Delta_1$ then the function $\psi(\Delta) = I_{II}(k_1 + \Delta\varkappa_1)$ reaches its minimum (its extreme). Then $\kappa_1 = B_1$. It follows from (9) that $\Delta_1 = -(A_1 / A_2)$.

The third and subsequent approximations are determined by applying analogous considerations. The evaluation of successive approximations can easily be programmed for a digital computer.

The described method can also be applied to evaluate the extreme of the I functional. Now the determination of κ_l [n - m, m, ϵ] is somewhat more involved (in the expression $\frac{dI}{d\Delta}\Big|_{\Delta=0} = \sum_{i=0}^{n+1} \frac{\partial I}{\partial I_i}\frac{dI_i}{d\Delta}\Big|_{\Delta=0}$ one has to evaluate also the derivatives $\frac{\partial\Phi}{\partial I_i}$ when $k = k_l$) and the procedure of finding the consecutive values $\Delta_1, \Delta_2, \ldots,$ becomes appreciably more difficult because of the dif-

ficulty of determining the extreme of the generally complex function $\xi(\Delta) = I(k_l + \Delta\kappa_l)$. However, in this case it is not indispensable to determine the values of the parameters $\theta_i = \theta_{i0}$ which were introduced only as an artifice. In each individual case one has to decide whether one should try to find directly the extreme of I or whether to find first the weighting function which will ensure the extreme of I_{II} at various values of the parameters θ_i and subsequently determine the values $\theta_i = \theta_{i0}$.

In the specific case when the functional I_{II} is given by (8), (9) and (10), A_1 and A_2 are expressed by means of k_1 and κ_1 as follows:

$$A_1 = \sum_{l=0}^{N}\Big\{\sum_{m=0}^{N} k_1[n-m, m, \varepsilon](\theta_1 R_z[n-m, n-l] + \theta_1 R_x[n-m, n-l] + x[n+l]x[n-m]) - \theta_1 HR_x[n-l, n+\varepsilon] - x[n-l]Hx[n,\varepsilon]\Big\}\varkappa_1[n-l, l, \varepsilon],$$

$$A_2 = \sum_{l=0}^{N}\sum_{m=0}^{N}(\theta_1 R_z[n-m, n-l] + \theta_1 R_x[n-m, n-l] + x[n-m]x[n-l])\times \varkappa_1[n-l, l, \varepsilon]\varkappa_1[n-m, m, \varepsilon],$$

$$\varkappa_1 = B_1 = \sum_{m=0}^{N} k_1[n-m, m, \varepsilon](\theta_1, R_z[n-m, m, \varepsilon] + \theta_1 R_x[n-m, m, \varepsilon] + x[n-l]x[n-m]) - \theta_1 HR_x[n-l, n, \varepsilon] - x[n-l]Hx[n, \varepsilon],$$

$$\Delta_1 = -\frac{A_1}{A_2} = -\frac{\sum_{l=0}^{N}\varkappa_1^2[n-l, l, \varepsilon]}{A_2} = -\frac{\sum_{l=0}^{N} B_1^2[n-l, l, \varepsilon]}{A_2}.$$

As a rule, the method of steepest descent implies a rapid convergence. In each case the speed of convergence depends on the starting element k_1 as well as on the form of the functional taken as the norm of the function k. The starting element k_1 should be taken such that its norm be finite.

E x a m p l e. Let useful signal X and interference Z be impressed on the input of a linear pulse system. The useful signal may be represented in the form $X = vt + X^\circ$, where X° is a random variable with a large variance, v a given quantity, and t the time.

The interference behaves as a stationary random process of normal distribution with the following characteristics: its mathematical expectation $M[Z] = 0$, its correlation function $R_z(\tau) = e^{-\alpha|\tau|}$. The variables X and Z are independent.

It is required to determine the weighting function of the pulse system with duration N = 1 of the transient process (the pulse sampling period is T) which would ensure maximum probability that the extrapolation error

$$E = X[n+\varepsilon+\varepsilon_e] - (X[n] + Z[n])k[n, 0, \varepsilon] - (X[n-1] + Z[n-1])k[n-1, 1, \varepsilon] \tag{21}$$

shall not exceed by numerical value a small quantity c. The probability of $|E| \leq c$, with c small, is found from the formula (1a).

The weighting function k selected is subjected to the restriction

$$k[n, 0, \varepsilon] + k[n-1, 1, \varepsilon] = 1, \tag{22}$$

which is feasible, since variable X° must have a large variance. It is known that the condition (22) produces nonstability of the xero order. Taking into account the initial data in our example and (22), we shall pick k in the class of stationary weighting functions. Consequently, $k = k[m, \epsilon]$ and (22) becomes

$$k[0, \varepsilon] + k[1, \varepsilon] = 1. \tag{22a}$$

Expression (21) can be rewritten in this case as

$$E = X^\circ + vT(n+\varepsilon+\varepsilon_e) - X^\circ - vTnk[0, \varepsilon] - vT(n-1)k[1, \varepsilon] - Z[n]k[0, \varepsilon] - Z[n-1]k[1, \varepsilon] = vT(\varepsilon+\varepsilon_e) + vTk[1, \varepsilon] - Z[n]k[0, \varepsilon] - Z[n-1]k[1, \varepsilon],$$

where $\epsilon_e T$ is the extrapolation time.

We determine a^2, σ^2, and I_{II}:

$$a^2 = (M[E])^2 = v^2T^2(\varepsilon+\varepsilon_e + k[1, \varepsilon])^2, \tag{23}$$

$$\sigma^2 = M[(E-a)^2] = k^2[0, \varepsilon] + 2e^{-\frac{\alpha}{T}}k[0, \varepsilon]k[1, \varepsilon] + k^2[1, \varepsilon], \tag{24}$$

$$I_{II} = \theta_1\sigma^2 + a^2 = \theta_1(k^2[0, \varepsilon] + 2e^{-\frac{\alpha}{T}}k[0, \varepsilon]k[1, \varepsilon] + k^2[1, \varepsilon]) + v^2T^2(\varepsilon+\varepsilon_e + k[1, \varepsilon])^2. \tag{25}$$

It is necessary to find function k_0 minimizing I_{II} under the condition (22a). This function should ensure the unconditional minimum of the functional

$$I' = I_{II} + 2\gamma_0(k[0, \varepsilon] + k[1, \varepsilon]), \tag{26}$$

where γ_0 is a Lagrange multiplier which can be determined from (22a). To determine k_0 we utilize (14) which in our case can be written as

$$\theta_1 k[0, \varepsilon] + \theta_1 e^{-\frac{\alpha}{T}} k[1, \varepsilon] = -\gamma_0,$$

$$\theta_1 e^{-\frac{\alpha}{T}} k[0, \varepsilon] + (\theta_1 + v^2T^2) k[1, \varepsilon] = -\gamma_0 - v^2T^2(\varepsilon + \varepsilon_e). \tag{27}$$

Solving the simultaneous equations (27) and (22a) we obtain

$$k_0[0, \varepsilon] = \frac{\theta_1 e^{-\frac{\alpha}{T}} [\gamma_0 + v^2T^2(\varepsilon + \varepsilon_e)] - \gamma_0(\theta_1 + v^2T^2)}{\theta_1\left(\theta_1 - \theta_1 e^{-2\frac{\alpha}{T}} + v^2T^2\right)},$$

$$k_0[1, \varepsilon] = \frac{\gamma_0\theta_1\left(e^{-\frac{\alpha}{T}} - 1\right) - v^2T^2(\varepsilon - \varepsilon_e)}{\theta_1\left(\theta_1 - \theta_1 e^{-2\frac{\alpha}{T}} + v^2T^2\right)}, \tag{28}$$

$$\gamma_0 = \frac{\theta_1\left[\theta_1\left(1 - e^{-2\frac{\alpha}{T}}\right) + v^2T^2\right] + v^2T^2(\varepsilon + \varepsilon_e)\left(1 - \theta_1 e^{-\frac{\alpha}{T}}\right)}{2\theta_1\left(e^{-\frac{\alpha}{T}} - 1\right) - v^2T^2}.$$

Formulas (28) enable one to determine the function k_0 when the values of ϵ and θ_1 are given. By keeping constant the value of the standardized time ϵ, one should find k_0 for several values of $\theta_1 \geq 0$ and determine subsequently the function $P = P(\theta_1)$; it is easy then to evaluate the value $\theta = \theta_{10}$ which corresponds to the maximum of P.

CONCLUSION

The presented method of finding the optimum pulse system ensuring the extreme of I seems to be more laborious when compared with the method of finding the pulse system ensuring the minimum of the mean-square error of the system, but basically, both methods are equally complex. The criterion I is sufficiently general; it can be utilized when selecting automatic control systems designed for various purposes. This method can also be used in the selection of optimum nonlinear systems [9].

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**See English translation.

THE EFFECT OF STATIONARY RANDOM PROCESSES ON AUTOMATIC CONTROL SYSTEMS CONTAINING ESSENTIALLY NONLINEAR ELEMENTS

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A method of calculating nonlinear automatic control systems under the action of stationary random processes is proposed. The method presented is a generalization of the method of statistical linearization and allows one to take into account the distortion of the spectrum of a random process by nonlinear elements.

At the present time the statistical calculation of automatic control systems containing essentially nonlinear elements is accomplished by the method of statistical linearization [1]. In essence this method consists in replacing a nonlinear element by an amplifier equivalent to it in some sense. The method is fairly simple and therefore is widely applicable in practice.

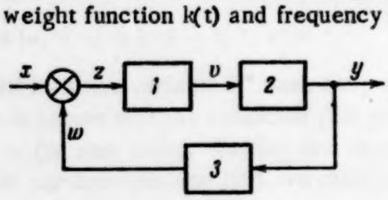
However, this method may never be considered completely justified, since the replacement of a nonlinear element by a linear amplifier does not take into consideration the distortion of the signal spectrum by the nonlinear element, and this distortion may in some cases prove to have a significant influence on the action of the system.

In this paper a more systematic approach to the problem of the statistical calculation of nonlinear systems is presented, in which the method of statistical linearization appears as the first approximation to the solution of the problem. As a consequence, one may estimate the applicability of the method of statistical linearization to a definite class of automatic control systems. The method presented may thus be considered a generalization and development of the ordinary method of statistical linearization.

1. Formulation of the Problem

We consider a system with feedback shown schematically in the figure. It contains one nonlinear element 1 with so-called standard characteristics [2]. All the remaining links are assumed to be linear with constant parameters. Any system containing one nonlinear element may be brought into such a form.

Suppose link 2 has weight function $g(t)$ and frequency characteristic $G(\omega)$, and the link consisting of 2 and 3 in series has weight function $k(t)$ and frequency characteristic $K(\omega)$.



We suppose the input to the system to be a Gaussian stationary random process $x(t)$; let m_x be the mathematical expectation, and σ_x^2 and $R_x(\tau)$ the variance and correlation function of the input process.

The problem consists in calculating the variance σ_y^2 and mathematical expectation m_y of the system's output process $y(t)$.

The process $z(t)$ represents the difference between the input process $x(t)$ and the process $w(t)$, which appears as a result of $z(t)$ acting through the nonlinear element and the succeeding linear elements. The presence of the nonlinear element leads to the fact that the distribution law for the random process at its output will necessarily not be Gaussian. However, the process is "normalized" by passing through the linear elements, so that the distribution law for the process $w(t)$ will be nearer to normal than the distribution law for the process at the output of the nonlinear element. Considering that the process $z(t)$ consists of the normal process $x(t)$ and the "normalized" process $w(t)$, one may assume that the distribution law $z(t)$ will be near normal. Proceeding from this, we make the assumption that the process $z(t)$ possesses a normal two-dimensional distribution law.

This assumption provides the possibility of solving the problem posed comparatively simply, since it is well known how normal fluctuations are transformed by a nonlinear element [2]. A similar assumption is made also in the method of statistical linearization.

2. Formation of the Basic Equation

Let $R_z(\tau)$ and $R_w(\tau)$ be correlation functions for processes $z(t)$ and $w(t)$, respectively, and $R_{zw}(\tau)$ and $R_{wz}(\tau)$, the mutual correlation functions for these processes.

We have the following equation:

$$R_x(\tau) = R_z(\tau) + R_w(\tau) + R_{zw}(\tau) + R_{wz}(\tau). \quad (1)$$

We shall find expressions for $R_w(\tau)$, $R_{zw}(\tau)$ and $R_{wz}(\tau)$ in terms of $R_z(\tau)$.

The normal two-dimensional distribution law for the process $z(t)$ at $m_z = 0$ may be put into the form of a series [3,4]

$$f(z, z_\tau) = \frac{1}{\sigma_z^2} \sum_{n=0}^{\infty} \frac{\rho_z^n(\tau)}{n!} \Phi^{(n+1)}\left(\frac{z}{\sigma_z}\right) \Phi^{(n-1)}\left(\frac{z_\tau}{\sigma_z}\right), \tag{2}$$

where $\rho_z(\tau) = R_z(\tau)/\sigma_z^2$, $\Phi(x)$ is the probability integral defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \tag{2'}$$

and $\Phi^{(n)}(x)$ is the nth derivative of this probability integral with respect to x.

Let the characteristic of the nonlinear element, that is, the connection between the output $v(t)$ and input $z(t)$ of the nonlinear element be given by the relation

$$v = F(z). \tag{3}$$

Characteristics for a different type of nonlinear element are introduced in [1].

Using (2) and (3), one may calculate the mathematical expectation and correlation function of the process $v(t)$ at the output of the nonlinear element [2]:

$$m_v = \int_{-\infty}^{\infty} F(z + m_z)\, d\Phi\left(\frac{z}{\sigma_z}\right), \tag{4}$$

$$R_v(\tau) = \sum_{n=1}^{\infty} \eta_n^2 \frac{\rho_z^n(\tau)}{n!}, \tag{5}$$

where

$$\eta_n = \int_{-\infty}^{\infty} F(z + m_z)\, d\Phi^{(n)}\left(\frac{z}{\sigma_z}\right). \tag{6}$$

The correlation function $R_w(\tau)$ may be written in the form

$$R_w(\tau) = \int_0^\infty \int_0^\infty k(\alpha) k(\beta) R_v(\tau - \alpha + \beta)\, d\alpha\, d\beta$$

or, considering (5),

$$R_w(\tau) = \sum_{n=1}^{\infty} \frac{\eta_n^2}{n!} \int_0^\infty \int_0^\infty k(\alpha) k(\beta) \rho_z^n(\tau - \alpha + \beta)\, d\alpha\, d\beta. \tag{7}$$

Analogously, for $R_{zw}(\tau)$ and $R_{wz}(\tau)$ we have the expressions

$$R_{zw}(\tau) = -\sigma_z \eta_1 \int_0^\infty k(\alpha) \rho_z(\tau + \alpha)\, d\alpha \tag{8}$$

and

$$R_{wz}(\tau) = -\sigma_z \eta_1 \int_0^\infty k(\alpha) \rho_z(\tau - \alpha)\, d\alpha. \tag{9}$$

Substituting (7), (8), and (9) into (1), we obtain an integral equation for $\rho_z(\tau)$:

$$\sigma_z^2 \rho_z(\tau) - \eta_1 \sigma_z \int_0^\infty k(\alpha) [\rho_z(\tau + \alpha) + \rho_z(\tau - \alpha)]\, d\alpha + \eta_1^2 \int_0^\infty \int_0^\infty k(\alpha) k(\beta) \rho_z(\tau - \alpha + \beta)\, d\alpha\, d\beta + \sum_{n=2}^{\infty} \frac{\eta_n^2}{n!} \int_0^\infty \int_0^\infty k(\alpha) k(\beta) \rho_z^n(\tau - \alpha + \beta)\, d\alpha\, d\beta = \sigma_x^2 \rho_x(\tau). \tag{10}$$

The equation obtained is nonlinear, since it contains a power of the function $\rho_z(\tau)$ higher than first. However, the nonlinear terms contain the factor $\eta_n^2/n!$ which, as an immediate calculation shows, is small in comparison to unity and is less that η_1^2. This circumstance allows us to apply the known method of expansion in powers of small parameters to the solution of (10). Such parameters in this case are $\eta_n^2/n!$. But it is not convenient to expand in powers of these parameters, since the number of them is infinite. Hence, it is more suitable by far to proceed in the following manner. We introduce the factor μ^n in front of the terms of the sums in the left portion of (10) and expand in powers of μ. From here on the solution proceeds according to the usual rules. Finally, one sets μ equal to one in the resulting expression. This may be done, because it is not necessary to consider μ to be a small quantity (in comparison with unity), since $\eta_n^2/n!$ are the small parameters.

3. Solution of the Integral Equation

Taking the above into consideration, we rewrite (10) in a somewhat modified form. We transfer the sum with the nonlinear terms to the right side and adjoin the factor μ^n before the terms of this sum:

$$\sigma_z^2 \rho_z(\tau) - \sigma_z \eta_1 \int_0^\infty k(\alpha) [\rho_z(\tau + \alpha) + \rho_z(\tau - \alpha)]\, d\alpha + \eta_1^2 \int_0^\infty \int_0^\infty k(\alpha) k(\beta) \rho_z(\tau - \alpha + \beta)\, d\alpha\, d\beta = \sigma_x^2 \rho_x(\tau) - \sum_{n=2}^{\infty} \mu^n \frac{\eta_n^2}{n!} \int_0^\infty \int_0^\infty k(\alpha) k(\beta) \rho^n(\tau - \alpha + \beta)\, d\alpha\, d\beta. \tag{11}$$

We represent the solution to (11) in the form of a series:

$$\rho_z(\tau) = \sum_{m=0}^{\infty} \mu^m \rho_m(\tau). \tag{12}$$

Substituting (12) into (11), we obtain

$$\sum_{m=0}^{\infty} \mu^m \Big\{ \sigma_z^2 \rho_m(\tau) - \eta_1 \sigma_z \int_0^{\infty} k(\alpha) [\rho_m(\tau+\alpha) + \rho_m(\tau-\alpha)]\, d\alpha + $$
$$+ \eta_1^2 \int_0^{\infty}\int_0^{\infty} k(\alpha) k(\beta) \rho_m(\tau - \alpha + \beta)\, d\alpha\, d\beta \Big\} = \sigma_x^2 \rho_x(\tau) - \tag{13}$$
$$- \sum_{n=2}^{\infty} \sum_{m_1, \ldots, m_n = 0}^{\infty} \mu^{n+m_1+m_2+\ldots+m_n} \frac{\eta_n^2}{n!} \int_0^{\infty}\int_0^{\infty} \rho_{m_1} \rho_{m_2} \rho_{m_3} \cdots \rho_{m_n} k(\alpha) k(\beta)\, d\alpha\, d\beta.$$

Now it is necessary to equate terms with equal powers of μ on the right and left sides. As a result we obtain a system of equations allowing us to determine successively the functions $\rho_m(\tau)$, expressing each in terms of the preceding. For concise notation we shall consider the left side of (11) as some linear integral operator L acting on the function $\rho_z(\tau)$:

$$L\rho_z(\tau) = \sigma_z^2 \rho_z(\tau) - \sigma_z \eta_1 \int_0^{\infty} k(\alpha) [\rho_z(\tau+\alpha) + \rho_z(\tau-\alpha)]\, d\alpha +$$
$$+ \eta_1^2 \int_0^{\infty}\int_0^{\infty} k(\alpha) k(\beta) \rho_z^n(\tau - \alpha + \beta)\, d\alpha\, d\beta. \tag{14}$$

Considering this, we obtain the following system of equations:

$$L\rho_0(\tau) = \sigma_x^2 \rho_x(\tau), \tag{15}$$

$$L\rho_1(\tau) = 0, \tag{16}$$

$$L\rho_2(\tau) = -\frac{\eta_2^2}{2!} \int_0^{\infty}\int_0^{\infty} k(\alpha) k(\beta) \rho_0^2(\tau - \alpha + \beta)\, d\alpha\, d\beta, \tag{17}$$

$$L\rho_3(\tau) = -\frac{\eta_3^2}{3!} \int_0^{\infty}\int_0^{\infty} k(\alpha) k(\beta) \rho_0^3(\tau - \alpha + \beta)\, d\alpha\, d_\beta, \tag{18}$$

$$L\rho_4(\tau) = -\frac{\eta_4^2}{4!} \int_0^{\infty}\int_0^{\infty} k(\alpha) k(\beta) \rho_0^4(\tau - \alpha + \beta)\, d\alpha\, d\beta - \frac{\eta_2^2}{2!} \int_0^{\infty}\int_0^{\infty} 2\rho_2\rho_0 k(\alpha)\, k(\beta)\, d\alpha\, d\beta, \tag{19}$$

$$L\rho_5(\tau) = -\frac{\eta_5^2}{5!} \int_0^{\infty}\int_0^{\infty} k(\alpha) k(\beta) \rho_0^5\, d\alpha\, d\beta -$$
$$- \frac{\eta_3^2}{3!} \int_0^{\infty}\int_0^{\infty} k(\alpha) k(\beta) 3\rho_0^2 \rho_2\, d\alpha\, d\beta - \frac{\eta_2^2}{2!} \int_0^{\infty}\int_0^{\infty} k(\alpha)\, k(\beta)\, 2\rho_0 \rho_3\, d\alpha\, d\beta. \tag{20}$$

This system of equations may be easily solved by Fourier analysis. If we introduce the notation

$$S_m(\omega) = \int_{-\infty}^{\infty} \rho_m(\tau) e^{-i\omega\tau} d\tau, \tag{21}$$

then, taking the Fourier transform of (15)-(20), and performing simple transformations, we obtain the following expressions for $S_m(\omega)$:

$$S_0(\omega) = \frac{1}{|\sigma_z - \eta_1 K(i\omega)|^2} \sigma_x^2 S_x(\omega), \tag{22}$$

$$S_1(\omega) \equiv 0, \tag{23}$$

$$S_2(\omega) = -\frac{|K(i\omega)|^2}{|\sigma_z - \eta_1 K(i\omega)|^2} \frac{\eta_2^2}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\nu) S_0(\omega - \nu)\, d\nu, \tag{24}$$

$$S_3(\omega) = -\frac{|K(i\omega)|^2}{|\sigma_z - \eta_1 K(i\omega)|^2} \frac{\eta_3^2}{3!} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} S_0(\nu_1) S_0(\nu_2) S_0(\omega - \nu_1 - \nu_2)\, d\nu_1 d\nu_2, \tag{25}$$

$$S_4(\omega) = -\frac{|K(i\omega)|^2}{|\sigma_z - \eta_1 K(i\omega)|^2} \Bigg[\frac{\eta_2^2}{2!} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 S_2(\nu) S_0(\omega - \nu)\, d\nu + \frac{\eta_4^2}{4!} \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} S_0(\nu_1) S_0(\nu_2) S_0(\nu_3) S_0(\omega - \nu_1 - \nu_2 - \nu_3)\, d\nu_1 d\nu_2 d\nu_3 \Bigg], \tag{26}$$

$$S_5(\omega) = -\frac{|K(i\omega)|^2}{|\sigma_z - \eta_1 K(i\omega)|^2} \Bigg[\frac{\eta_2^2}{2!} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 S_0(\nu) S_3(\omega - \nu)\, d\nu + \frac{\eta_3^2}{3!} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} 3 S_0(\nu_1) S_0(\nu_2) S_2(\omega - \nu_1 - \nu_2)\, d\nu_1 d\nu_2 + \frac{\eta_5^2}{5!} \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} S_0(\nu_1) S_0(\nu_2) S_0(\nu_3) S_0(\nu_4) S_0(\omega - \nu_1 - \nu_2 - \nu_3)\, d\nu_1 d\nu_2 d\nu_3 d\nu_4 \Bigg]. \tag{27}$$

. . . .

Here $K(i\omega)$ is the frequency characteristic of the serially joined links 2 and 3 (see the figure), corresponding to the weight function $k(t)$.

Considering that

$$\rho_m(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_m(\omega) e^{i\omega\tau} d\omega. \tag{28}$$

and also bearing (12) in mind, we obtain for $\rho_z(\tau)$ the expression

$$\rho_z(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} S_m(\omega) e^{i\omega\tau} d\omega. \tag{29}$$

The functions $S_m(\omega)$ entering into (29) depend on the values m_z and σ_z, which are unknown beforehand. It is necessary to calculate them. We obtain the first equation for the determination of m_z and σ_z from (29):

$$\sigma_z^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_z^2 \sum_{m=0}^{\infty} S_m(\omega)\, d\omega. \tag{30}$$

Since the mathematical expectation m_x of the input process is constant, one may write the following equation for the mathematical expectation of the process $z(t)$:

$$m_z = \frac{m_x}{1 + k_0 \chi(m_z, \sigma_z)}. \tag{31}$$

where k_0 is the amplification coefficient of the system, and $\chi = m_y/m_z$ is determined by (4). Equation (31) is a second equation for determining m_z and σ_z. Similar equations are also obtained by the method of statistical linearization. For systems whose linear parts are described by differential equations of higher order, the following method of solution of (30) and (31) is usually employed.

For the initial values of m_z and σ_z one may take any quantities, let us say m_x and σ_x. Setting them into the right sides of (30) and (31), we obtain m_{z1} and σ_{z1}, in general different from m_{z0} and σ_{z0}. Then the values m_{z1} and σ_{z1} obtained should again be substituted into the right sides of (30) and (31), after which the new m_{z2} and σ_{z2} are obtained. This process is to be continued until upon a substitution, results are obtained which coincide with zero-order accuracy with the results of the preceding substitution. We note that the procedure of substitution consists in calculating the functions $S_m(\omega)$ from (22) - (27) for given values of m_z and σ_z, thereafter calculating (30), as well as (31). Practical calculation by the method of statistical linearization shows that a similar procedure of solving the equation allows a sufficiently rapid calculation of m_z and σ_z.

After m_z and σ_z are calculated, and also $\eta_1(m_z, \sigma_z)$, $\eta_2(m_z, \sigma_z)$ and so on, the variance of the process y(t) at the output is easily calculated, by using (7) for $\tau = 0$, and substituting the weight function $g(\omega)$ of link 2 in the straight circuit, for $k(\omega)$. Taking the Fourier transform, we obtain the following relation:

$$\sigma_y^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)|^2 \Big\{\eta_1^2 S_z(\omega) + \frac{\eta_2^2}{2!} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_z(\nu) S_z(\omega - \nu)\, d\nu + \frac{\eta_3^2}{6} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} S_z(\nu_1) S_z(\nu_2) S_z(\omega - \nu_1 - \nu_2)\, d\nu_1\, d\nu_2 + \ldots\Big\}. \tag{32}$$

where $S_z = \sum_{m=0}^{\infty} S_m(\omega)$ for the calculated values m_z and σ_z.

We return to (22). It represents an equation for the calculation of the spectral density of the process at the input to the nonlinear element by the method of statistical linearization. Hence, the method of statistical linearization is obtained as the first approximation to a more complete solution to the problem. The functions $S_m(\omega)$ for $m > 0$ are further approximations. Since they contain the parameters $\eta^2/2!$, $\eta^2/3!$, etc., which under definite conditions are negligible, the further approximations under these conditions carry but small contributions in forming the spectral density, and the method of statistical linearization gives the correct solution. However, there may occur conditions under which one may not neglect the further approximations, and then the method of statistical linearization yields an invalid solution. This may occur, for example, when the spectrum of the signal x(t) is essentially concentrated in a neighborhood of the frequency ω_0, and the linear link has a resonance at the frequency $2\omega_0$ or $3\omega_0$, and so on. In order to clarify these conditions, it is necessary to carry out numerical calculations for different specific cases. Such a calculation may be accomplished simply with contemporary computing machines.

CONCLUSION

In this paper it is shown that under the assumption on normal distribution of a random process at the input to a nonlinear element, the problem of calculating the dispersion at the output of a system with feedback containing a nonlinear element may be solved more accurately than may be done using only the method of statistical linearization.

The method of statistical linearization is obtained as the first approximation in the approach to the solution of the problem presented, and consequently, the possibility arises of being able to test its applicability in different cases.

The calculations by the formulas given may be carried out with the aid of contemporary computational techniques.

In conclusion, I take this opportunity to thank R. L. Stratonovich deeply for valuable counsel and direction given by him in connection with this paper.

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AN INTERPOLATION METHOD FOR ANALYZING AUTOMATIC CONTROL SYSTEM ACCURACY UNDER RANDOM STIMULI

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An approximate method is suggested for constructing the solution of a system of ordinary differential equations containing random functions and random variables, and formulas are derived for the computation of the moments. The method is a development of certain ideas of Chebyshev [1].

INTRODUCTION

In designing or, when necessary, in improving an automatic control system, the problem arises of analyzing the accuracy of control when one takes into account random disturbances acting on the system.

The systems of differential equations which describe the operation of an automatic control system with random stimuli are, in the general case, nonlinear:

$$\frac{dY_i}{dt} = F_i\left(t, Y_1, Y_2, \ldots, Y_n, X_1(t), X_2(t), \ldots, X_l(t), \lambda_1, \lambda_2, \ldots, \lambda_s\right), \qquad (1)$$
$$t = 0, \quad Y_i(0) = y_{i0} \quad (i = 1, 2, \ldots, n).$$

where the Y_i are parameters characterizing the control process, $X_1(t), X_2(t), \ldots, X_l(t)$ are random functions of time whose probability characteristics are assumed to be given, and $\lambda_1, \lambda_2, \ldots, \lambda_s$ are random variables whose probability characteristics are also assumed to be given.

The initial conditions ($t = 0$, $Y_i(0) = y_{i0}$) are also random variables with known probability characteristics. The functions F_i ($i = 1,2,\ldots,n$) are such that there exists a unique solution of system (1) for each choice of random initial conditions, random variables $\lambda_1,\ldots,\lambda_s$ and random functions $X_1(t), X_2(t),\ldots,X_l(t)$.

Ordinarily, to analyze accuracy of a control system, the nonlinear differential equations are replaced, with certain assumptions, by linear ones, and then one uses the quite well developed methods for analyzing the accuracy of linear systems with random stimuli present [2,3]. Of all the existing linearization methods, the most practical in the given case is Kazakov's statistical linearization method [4,5].

In [6,7], for the analysis of accuracy of nonlinear systems with random stimuli, there were presented quite general theoretical methods which could serve as starting points for the construction of engineering methods and, in the simplest cases, for engineering calculations as well. Thus, in [7], Dostupov put forward an engineering method for analyzing the accuracy of nonlinear control systems with random stimuli, this method being a development of Pugachev's method which assumes the use of the latest computing technology.

In the present work we present a new approximate method for analyzing the accuracy of an automatic control system with random stimuli, wherein the use of electronic computers is also assumed, but where the method is based on other principles than those of [6,7] and does not require that one solve an auxiliary system of algebraic equations, as is the case in [7].

1. Posing of the Problem

If we use the canonical expansions of the random functions $X_1(t), X_2(t),\ldots,X_l(t)$ and limit ourselves to a finite number of terms in these expansions, we may approximately replace system (1) of differential equations by a system of the form

$$\frac{dY_i}{dt} = F_i(t, Y_1, Y_2, \ldots, Y_n, \lambda_1, \lambda_2, \ldots, \lambda_m). \qquad (2)$$
$$t = 0, \quad Y_i(0) = y_{i0} \quad (i = 1, 2, \ldots, n).$$

which contains only random variables in the right members, thus significantly simplifying the investigation.

All the assumptions made relative to (1) remain in force for this new system. Moreover, without reducing the generality, one may assume that the initial conditions are not random, since otherwise one could attain such a position by a linear change of variables, which would only entail an increase in the number of random variables entering the right members of (2).

The random variables $\lambda_1, \lambda_2, \ldots \lambda_m$ will be assumed to be independent, which is also not a very rigid requirement, since in many practical cases one may attain this state by means of a linear transformation of these quantities. As the probability characteristics of the random variables, there are given the distribution densities $P_1(\lambda_1), P_2(\lambda_2), \ldots, P_m(\lambda_m)$, for which there exist moments of every order, i.e.,

$\int \lambda_j^k P_j(\lambda_j)\, d\lambda_j = m_j^{(k)}$, where the $m_j^{(k)}$ are certain constants for any k and j ($k = 0,1,2,\ldots$; $j = 1,2,\ldots,m$).

Since, by assumption, there exists a unique solution of (2) for each choice of the random variables $\lambda_1, \lambda_2, .., \lambda_m$, which are independent parameters of this system, then the solution of system (2) can be given in the form of certain functions of time and the random variables $\lambda_1, \lambda_2, .., \lambda_m$:

$$Y_i = Y_i(t, \lambda_1, \lambda_2, \ldots, \lambda_m) \qquad (i = 1, 2, \ldots, n). \quad (3)$$

For the set of functions in (3), which are solutions of (2) for any values of the random variables $\lambda_1, \ldots, \lambda_m$, we shall assume the existence of the second moments, i.e.,

$$M[Y_i^2] = \int_\Delta \ldots \int Y_i^2(t, \lambda_1, \lambda_2, \ldots, \lambda_m) \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j = K_i \qquad (i = 1, 2, \ldots, n), \quad (4)$$

where the K_i are constants and Δ is the range of values of the random variables $\lambda_1, \ldots, \lambda_m$, and is an m-dimensional finite or infinite parallelopiped.

We choose some set of numbers $\{\mu_{1k_1}, \mu_{2k_2}, \ldots, \mu_{mk_m}\}$ ($k_1 = 0, 1, 2, \ldots, q$; $k_2 = 0, 1, 2, \ldots, q$; $k_m = 0, 1, 2, \ldots, q$) which represents $(q+1)^m$ points in region Δ.

Let a known solution of (1) be computed at each of the aforementioned points:

$$y_i(t, \mu_{1k_1}, \mu_{2k_2}, \ldots, \mu_{mk_m}) \qquad (i = 1, 2, \ldots, n) \quad (5)$$

$$(k_1 = 0, 1, 2, \ldots, q;\ k_2 = 0, 1, 2, \ldots, q;\ \ldots;\ k_m = 0, 1, 2, \ldots, q).$$

It is required, from the given set (5) of solutions of (2), to construct the functions

$$Y_{iq}(t, \lambda_1, \lambda_2, \ldots, \lambda_m) \qquad (i = 1, 2, \ldots, n), \quad (6)$$

which are sufficiently accurate approximations to the solution and, from the functions in (6), to construct the simplest possible formulas for computing the solution's probability characteristics.

2. Probability Principle for Constructing Approximate Solutions

One of the most general principles for construction of the functions in (6) is the principle of minimum probability for each fixed q.

In the case under consideration, this principle may be formulated as follows: from among the set of all the functions $\widetilde{Y}_{iq}(t, \lambda_1, \lambda_2, \ldots, \lambda_m)$, constructed on the basis of the functions $y_i(t, \mu_{1k_1}, \mu_{2k_2}, \ldots, \mu_{mk_m})$, we choose a function such that

$$P[\,|\widetilde{Y}_{iq} - Y_i(t, \lambda_1, \lambda_2, \ldots, \lambda_m)| > \varepsilon] = \min \qquad (i = 1, 2, \ldots, n) \quad (7)$$

for each fixed q, where $\epsilon > 0$ is any arbitrarily small given number; the symbol P[...] denotes the probability that the inequality within the square brackets actually holds.

If the probability in (7) tends to zero as q increases, one has convergence in probability, which is ordinarily written in the form of a limiting equality,

$$\lim_{q\to\infty} P[\,|\widetilde{Y}_{iq}(t, \lambda_1, \lambda_2, \ldots, \lambda_m) - Y_i(t, \lambda_1, \lambda_2, \ldots, \lambda_m)| > \varepsilon] = 0 \quad (i = 1, 2, \ldots, n). \quad (8)$$

Instead of the minimum probability principle, one may use the computationally simpler principle of the minimum of the second moment of the difference between the exact and the approximate solutions:

$$M[(\widetilde{Y}_{iq} - Y_i)^2] = \int_\Delta \ldots \int (\widetilde{Y}_{iq} - Y_i)^2 \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j = \min \qquad (i = 1, \ldots, n). \quad (9)$$

Indeed, according to Chebyshev's inequality,

$$P[\,|\widetilde{Y}_{iq} - Y_i| > \varepsilon] < \frac{M[(\widetilde{Y}_{iq} - Y_i)^2]}{\varepsilon^2}. \quad (10)$$

It follows from Chebyshev's inequality that, if

$$\lim_{q\to\infty} M[(\widetilde{Y}_{iq} - Y_i)^2] = 0, \quad (11)$$

then there will be convergence in probability.

If the function $\widetilde{Y}_{iq}$ is chosen from the class of all possible m-dimensional polynomials of degree q, then the problem of minimizing the second moment is, in theory, easily solved.

Indeed, by virtue of the assumptions as to the weight functions, for each function $P_j(\lambda_j)$ ($j = 1, 2, .., m$) there exists [9] a system, unique to within a common factor, of orthogonal polynomials $H_{j0}, H_{j1}, H_{j2}, \ldots$, i.e., such that

$$\int_{a_j}^{b_j} H_{ji}(\lambda_j) H_{jk}(\lambda_j) P_j(\lambda_j)\, d\lambda_j = \begin{cases} 0 & \text{for } i \neq k, \\ C_{ij}^2 & \text{for } i = k, \end{cases}$$

where the numbers a_j and b_j can be infinite.

Let

$$Y_i^{(F)} = \sum_{n_1 n_2 \ldots n_m} a_{in_1 n_2 \ldots n_m}(t) \prod_{j=1}^{m} H_{jn_j}(\lambda_j), \quad (12)$$

where

$$a_{in_1 n_2 \ldots n_m}(t) = \frac{\int_\Delta \ldots \int Y_i \prod_{j=1}^{m} H_{jn_j}(\lambda_j) P_j(\lambda_j)\, d\lambda_j}{\int_\Delta \ldots \int \prod_{j=1}^{m} H_{jn_j}^2(\lambda_j) P_j(\lambda_j)\, d\lambda_j} \quad (13)$$

$$(n_1, n_2, \ldots, n_m = 0, 1, 2, \ldots).$$

From the theory of Fourier series we know that the minimum of the integral in (9) in the class of all possible m-dimensional polynomials of degree q is furnished by segments of the Fourier series for the functions Y_i ($i = 1, 2, .., n$), i.e., the polynomials

$$Y_{iq}^{(F)} = \sum_{n_1 n_2 \dots n_m} a_{in_1 n_2 \dots n_m}(t)$$

$$\prod_{j=1}^{m} H_{jn_j}(\lambda_j) \begin{pmatrix} n_1 = 0, 1, 2, \dots, q \\ n_2 = 0, 1, 2, \dots, q \\ \cdots\cdots\cdots\cdots \\ n_m = 0, 1, 2, \dots, q \end{pmatrix}. \qquad (14)$$

However, the problem we have posed can be considered solved only theoretically since, for the computation of the m-fold integrals entering into (14), there are only $(q+1)^m$ points at which the values of the integrands may be assumed to be known.

If the solution of (2) can be presented exactly in the form of m-dimensional polynomials in the random variables $\lambda_1, \lambda_2, \dots, \lambda_m$ of degrees not exceeding the number q, then the problem as posed can be solved exactly by means of Lagrange interpolating polynomials.

In fact, in this case, from the given $(q+1)^m$ interpolation nodes, one can construct, using the Lagrange interpolation formula for the m-dimensional case [8], an m-dimensional polynomial of degree not exceeding q:

$$Y_{iq} = \sum_{k_1 k_2 \dots k_m} y_i(t, \mu_{1k_1}, \mu_{2k_2}, \dots, \mu_{mk_m}) \qquad (15)$$

$$\prod_{j=1}^{m} \frac{\omega_{j,q+1}(\lambda_j)}{\omega'_{j,q+1}(\mu_{jk_j})(\lambda_j - \mu_{jk_j})}$$

$$(i = 1, 2, \dots, n)\ (k_1 = 0, 1, 2, \dots, q;$$

$$k_2 = 0, 1, 2, \dots, q; \dots; k_m = 0, 1, 2, \dots, q).$$

coinciding identically with the solution of (2), where

$$\omega_{j,q+1}(\lambda_j) =$$

$$= (\lambda_j - \mu_{1k_j})(\lambda_j - \mu_{2k_j}) \dots (\lambda_j - \mu_{mq_j}); \quad \omega'_{j,q+1}(\mu_{jk_j})$$

is the derivative, computed for $\lambda_j = \mu_{jk_j}$.

However, the case more frequently arises in the applications when the solution can be given in the form of m-dimensional polynomials of degree q only approximately, and the interpolation polynomials of (15), therefore, will in general be only approximations whose closeness, in some sense or other, to the solution of (2) depends on the choice of the interpolation nodes.

Since the interpolation nodes can be chosen arbitrarily, it is then necessary to so manage this choice that the second moments of the difference between the solution and the approximation to it in the form of interpolation polynomials be, for each fixed q, as close as possible to their minimum values attainable in the class of all possible m-dimensional polynomials of degree q, and so that the approximation, in the form of interpolation polynomials, converge in probability to the exact solution of (2).

If one chooses as the $(q+1)^m$ interpolation nodes all possible combinations of roots of the orthogonal polynomials $H_{j,q+1}(\lambda_j)(j = 1, 2, \dots, n)$ of degree $q+1$, which may always be done since the roots of these polynomials are real, different and lie in the intervals of orthogonality, then these requirements can, to a known degree, be satisfied [9].

In this case, the interpolation polynomials which are the approximate solution may be written in the form

$$\widetilde{Y}_{iq} = \sum_{k_1 k_2 \dots k_m} y_i(t, \mu_{1k_1}, \mu_{2k_2}, \dots, \mu_{mk_m})$$

$$\prod_{j=1}^{m} \frac{H_{j,q+1}(\lambda_j)}{H'_{j,q+1}(\mu_{jk_j})(\lambda_j - \mu_{jk_j})} \qquad (16)$$

$$(i = 1, 2, \dots, n) \quad (k_1 = 0, 1, 2, \dots, q;$$

$$k_2 = 0, 1, 2, \dots, q; \dots; k_m = 0, 1, 2, \dots, q);$$

$$H_{j,q+1}(\lambda_j) = (\lambda_j - \mu_{1kj})(\lambda_j - \mu_{2kj}) \dots (\lambda_j - \mu_{mkj});$$

$H_{j,q+1}(\mu_{jk_j})$ is the derivative, computed for $\lambda_j = \mu_{jk_j}$; the numbers μ_{jk_j} satisfy the algebraic equations

$$H_{j,q+1}(\mu_{jk_j}) = 0 \qquad (j = 1, 2, \dots, m).$$

It can be shown that the construction of the interpolation polynomial by (16) with the thus chosen interpolation nodes, which we shall for brevity henceforth call Chebyshev type nodes, is equivalent to the construction of the segment of the Fourier series from the $(q+1)^m$ first terms if, in computing the coefficients of the expansion, one uses a Gauss type formula for mechanical quadrature corresponding to weights equal to the given probability density functions. Since Gauss type mechanical quadrature formulas are the quadrature formulas with the highest degree of algebraic accuracy then, in general, it is impossible to give a set of $(q+1)^m$ interpolation nodes from which one could construct an m-dimensional polynomial of degree q lying closer to the segment of Fourier series (14), from the point of view of minimizing the integral in (9), than the polynomial constructed from the Chebyshev type nodes.

As was established earlier [cf. (11)], to prove convergence in probability it suffices to establish a limiting equality to zero of the second moment of the difference between the exact and the approximate solutions.

In the given case,

$$\lim_{q\to\infty} \int \dots \int_{\Delta} (Y_i - \widetilde{Y}_{iq})^2 \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j =$$

$$= \int \dots \int_{\Delta} Y_j^2 \prod_{j=1}^{m} P_i(\lambda_j)\, d\lambda_j -$$

$$- 2 \lim_{q\to\infty} \int \dots \int_{\Delta} \widetilde{Y}_i Y_{iq}$$

$$\prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j + \lim_{q\to\infty} \int \dots \int_{\Delta} \widetilde{Y}_{iq}^2 \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j = 0, \qquad (17)$$

if

$$\lim_{q\to\infty} \int \dots \int_{\Delta} \widetilde{Y}_{iq}^2 \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j = \int \dots \int_{\Delta} Y_i^2 \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j \qquad (18)$$

and

$$\lim_{q\to\infty} \int \dots \int_{\Delta} Y_i \widetilde{Y}_{iq} \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j = \int \dots \int_{\Delta} Y_i^2 \prod_{j=1}^{m} P_j(\lambda_j)\, d\lambda_j. \qquad (19)$$

By substituting, in (18), the expanded expression for the interpolation polynomial, and by taking into account the orthogonality of the polynomials $H_{j,q+1}(\lambda_j)$, we obtain, after a number of transformations,

$$\lim_{q\to\infty}\int_\Delta\cdots\int\widetilde{Y}_{iq}^2\prod_{j=1}^m P_j(\lambda_j)\,d\lambda_j = \tag{20}$$

$$=\lim_{q\to\infty}\sum_{k_1k_2\ldots k_m} y_i(t,\mu_{1k_1},\mu_{2k_2},\ldots,\mu_{mk_m})\prod_{j=1}^m \rho_{jk_j}$$

$$(i=1,2,\ldots,n)\ (k_1=0,1,2,\ldots,q;$$

$$k_2=0,1,2,\ldots,q;\ldots;\quad k_m=0,1,2,\ldots,q),$$

where $\rho_{jk_j}=\int_{a_j}^{b_j}\frac{H_{j,q+1}(\lambda_j)(P_j(\lambda_j)\,d\lambda_j}{H'_{j,q+1}(\mu_{jk_j})(\lambda_j-\mu_{jk_j})}$ are Christoffel numbers.

Formula(20) is a Gauss type mechanical quadrature formula with weights $\prod_{j=1}^m P_j(\lambda_j)$ for computing the integral $\int_\Delta\cdots\int Y_i^2\prod_{j=1}^m P_j(\lambda_j)\,d\lambda_j$.

It is well known* that the mechanical quadrature process converges for any Riemann-integrated function if the domain of integration is bounded. If the region is unbounded the same state of affairs can be attained if, in constructing the interpolation polynomial, one does not take into consideration nodes which lie at a sufficiently great distance from the origin of coordinates, a process which can only have a beneficial effect in calculations with electronic computers (since cases of overflow are avoided).

Since (19) follows from (18), it can be asserted that, when the solution of (2) is such that the second moments in (4) exist, then the interpolation process constructed on Chebyshev type nodes tends in probability to the solution of this system.

3. Calculated Formulas for Computing the Moments

The approximate formulas for computing the moments are obtained from the ordinary formulas expressing these moments in general form if, in these formulas, the exact solution of (2) is replaced by approximate ones in the form of interpolation polynomials constructed from Chebyshev type nodes.

We now obtain the formulas for computing the first- and second-order moments:

$$M[Y_i(t,\lambda_1,\lambda_2,\ldots,\lambda_m)]=\int_\Delta\cdots\int Y_i\prod_{j=1}^m P_j(\lambda_j)\,d\lambda_j\approx$$

$$\approx\int_\Delta\cdots\int\widetilde{Y}_{iq}\prod_{j=1}^m P_j(\lambda_j)\,d\lambda_j\qquad(i=1,2,\ldots,n).$$

$$M[Y_i(t_1,\lambda_1,\lambda_2,\ldots,\lambda_m)Y_k(t_2,\lambda_1,\lambda_2,\ldots,\lambda_m)]\approx$$

$$\approx\int_\Delta\cdots\int\widetilde{Y}_{iq}(t_1,\lambda_1,\lambda_2,\ldots,\lambda_m)\widetilde{Y}_{kq}(t_2,\lambda_1,\lambda_2,\ldots,\lambda_m)\prod_{j=1}^m P_j(\lambda_j)\,d\lambda_j \tag{21}$$

$$(i=1,2,\ldots,n;\quad j=1,2,\ldots,n).$$

By substituting in these formulas the expanded expressions for $\widetilde{Y}_{iq}$ and $\widetilde{Y}_{iq}$ and carrying out a number of transformations, we get

$$M[Y_i]\approx\sum_{k_1k_2\ldots k_m} y_i(t,\mu_{1k_1},\mu_{2k_2},\ldots,\mu_{mk_m})\prod_{j=1}^m\rho_{jk_j}.$$

$$M[Y_i(t_1,\lambda_1,\lambda_2,\ldots,\lambda_m)Y_k(t_2,\lambda_1,\lambda_2,\ldots,\lambda_m)]\approx$$

$$\approx\sum_{k_1k_2\ldots k_m} y_i(t,\mu_{1k_1},\mu_{2k_2},\ldots,\mu_{mk_m})$$

$$y_k(t_2,\mu_{1k_1},\mu_{2k_2},\ldots,\mu_{mk_m})\prod_{j=1}^m\rho_{jk_j}$$

$$(i=1,2,\ldots,n;\quad j=1,2,\ldots,n)$$

$$(k_1=0,1,2,\ldots,q;\quad k_2=0,1,2,\ldots,q;\ldots;$$

$$k_m=0,1,2,\ldots,q).$$

where the ρ_{jk_j} are Christoffel numbers.

The approximate formulas for computing higher-order moments have an analogous form.

In computing the moments of the solution of (2) by (21), it is most convenient to determine the functions $y_i(t,\mu_{1k_1},\mu_{2k_2},\ldots,\mu_{mk_m})$ $(i=1,2,\ldots,n)$ by numerical methods, carrying out the computations on electronic digital computers, since the accuracy of analog computers is, in many cases, inadequate; moreover, the final computations by (21) equally require discrete (digital) technology. The Chebyshev type nodes and the Christoffel numbers $\rho_{jk_1}, \rho_{jk_2},\ldots\rho_{jk_m}$ are computed from the given probability density functions. However, in many practical cases this is not necessary, since one can use prepared tables [10] of roots of orthogonal polynomials and the Christoffel numbers corresponding to these polynomials.

For example, let the random variable λ_j be distributed normally, $P_j(\lambda_j)=\frac{1}{\sqrt{2\pi}\sigma_{\lambda_j}}\exp\left[-\frac{(\lambda_j-\bar{\lambda}_j)^2}{2\sigma_{\lambda_j}^2}\right]$. We set

*From the Steklov-Polya theorem.

$\lambda_j = \bar{\lambda}_j + x\sigma_{\lambda_j}\sqrt{2}$. The distribution density of the random variable $\underline{x}$ will be $P(x) = e^{-x^2}/\sqrt{\pi}$. The Chebyshev-Hermite polynomials

$$H_n(x) = \left(-\frac{1}{2}\right)^n e^{x^2}\frac{d^n}{dx^n}[e^{-x^2}]$$

form a system of polynomials which are orthogonal on the interval $(-\infty, \infty)$.

Let $x_{k,\,q+1}^{q+1}$ $(k = 0, 1, 2, \ldots, q)$ be the roots of the Chebyshev-Hermite polynomial of degree q + 1. In this case, the Chebyshev type interpolation nodes and the Christoffel numbers are computed by the formulas

$$\lambda_{jk_j} = \bar{\lambda}_j + x_k^{(q+1)}\sigma_{\lambda_j}\sqrt{2}$$

$$(k = 0, 1, 2, \ldots, q),$$

$$\rho_{jk_j} = \int\limits_{-\infty}^{\infty} \frac{H_{q+1}(x)\,P(x)\,dx}{H'_{q+1}(x_k^{(q+1)})(x - x_k^{(q+1)})} = \rho_k^{(q+1)} \qquad (22)$$

$$(k = k_j).$$

For uniform distributions on the intervals $[a_j, b_j]$, the Chebyshev type nodes and the Christoffel numbers are computed by the formulas

$$\lambda_{jk_j} = \frac{b_j - a_j}{2}x_k^{(q+1)} + \frac{b_j + a_j}{2}$$

$$(k = 0, 1, 2, \ldots, q), \qquad (23)$$

$$\rho_{jk_j} = \rho_k^{(q+1)} = \int\limits_{-1}^{1}\frac{U_{q+1}(x)\,dx}{U'_{q+1}(x_k^{(q+1)})(x - x_k^{(q+1)})},$$

where $U_{q+1}(x) = \frac{(q+1)!}{(2q+2)!}\frac{d^{q+1}}{dx^{q+1}}(x^2 - 1)^{q+1}$ is the Legendre polynomial of degree q + 1 and the (k = 0, 1, 2,..., q) are its roots.

Since the method proposed here, for q = 1, gives a linear approximation to the solution of (2), it can be used in that case as one of the possible linearization methods which does not require that the equations of system (2) be transformed.

CONCLUSIONS

The method presented here can be used in the investigation of the accuracy of operation of an automatic control system described by a system of ordinary differential equations for which, with given initial conditions and for any choice of the random variables entering into the system's right members, there exists a unique solution with finite second moments. Thus, the method can also be used for the investigation of systems with essentially nonlinear characteristics.

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Example. An automatic control system is described by equations of the form

$$\frac{dY}{dt} + Y^2 = (\lambda + 1)^2, \quad t = 0, \quad Y(0) = 0,$$

where λ is a random variable with zero mathematical expectation, and dispersion $M[\lambda^2] = 0.01$;

$$P(\lambda) = \frac{1}{\sqrt{2\pi}\,0.1}e^{-\frac{\lambda^2}{0.02}}.$$

It is required to determine the mathematical expectation and dispersion of the output coordinate Y.

By using (21) and (22) and making use of tables [10], we find that, for q = 1, M[Y] = 0.949 and D[Y] = 0.0164, while for q = 2, M[Y] = 0.964 and D[Y] = 0.0123.

The exact solution, obtained by solving the given equation with the given initial conditions, is that

M[Y] = 0.963, D[Y] = 0.0123.

Thus, in the given example, for q = 2 the approximate values of the mathematical expectation and the dispersion, computed by the interpolation method, coincide with the exact values to within one unit in the third decimal place.

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THEORY OF IDEAL MODELS OF AN EXTREMAL CONTROLLER*

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By means of the theory of point transformations, an analysis is made of the operation of an extremal control system with a commutator. In the discussion, no essential limitations are imposed on the form of the nonlinearities. The existence of a limiting cycle is proven, and the character of its establishment is determined; the possibility of multiple oscillations, and the conditions for their generation, are investigated. The effect of asymmetry on the extremal function is determined, and an estimate of the accuracy of controller operation is made.

Existing automatic control systems solve the problem of maintaining a controlled quantity either constant or else changing in accordance with a previously prescribed law (in the case of so-called "programmed" control).

After a disturbing factor has acted, the controlled quantity changes its value, and the automatic controller must, by means of its action on the controlling parameter, take it to some given level in some interval of time.

If the problem to be solved by the automatic controller consists in maintaining a given program, then the control system just sketched is theoretically valid for its solution. Thus, for example, we have the problems of maintaining the number of rotations of a machine's principal shaft, steam or gas pressure in a trunk, voltage or current in a cii cuit, the problem of maintaining a given ratio of components, etc.

However, as technology developed, a number of problems arose which differed in principle from those just cited, these being the problems of the automatic search for, and maintaining of, extremal values of the controlled quantity, either at its upper or its lower boundary.

Such problems arise in the most diverse domains of technology (aviation, motor design, etc.).

Naturally, such problems can be solved only approximately by the ordinary control systems.

Several years ago the author of the present paper developed the automatic control principle which solves these problems, the principle which one may call that of extremal control.

Some of the questions in the theory of operation of extremal controllers were already considered previously by the author [1,2].

In the present paper, using the methods of point transformation theory, we investigate the operation of one of the extremal control schemes.

1. Principle of Operation of the Extremal Controller

In view of the fact that the form of the functional relationship between the controlled and the controlling quantities is not known and, moreover, varies in the course of time by a completely unknown law, to give this relationship in advance is not possible.

It is in relation to this that the idea of introducing an artificial disturbance is made basic to the principle of operation of the extremal controller.

All the motions arising in the system as a result of this disturbance are divided into two classes: a) the class of admissible motions, b) the class of inadmissible motions.

To the class of admissible motions are assigned all those which move the controlled quantity closer to the required extremum or boundary, while the motions which take the controlled quantity further away are assigned to the class of inadmissible motions.

By means of a special device, the direction of the process is verified, and if this direction causes the controlled quantity to approach the required extremum, then the disturbance is accepted; if the quantity moves away from the extremum, then the sign of the disturbance is reversed, and motion in the required direction then commences.

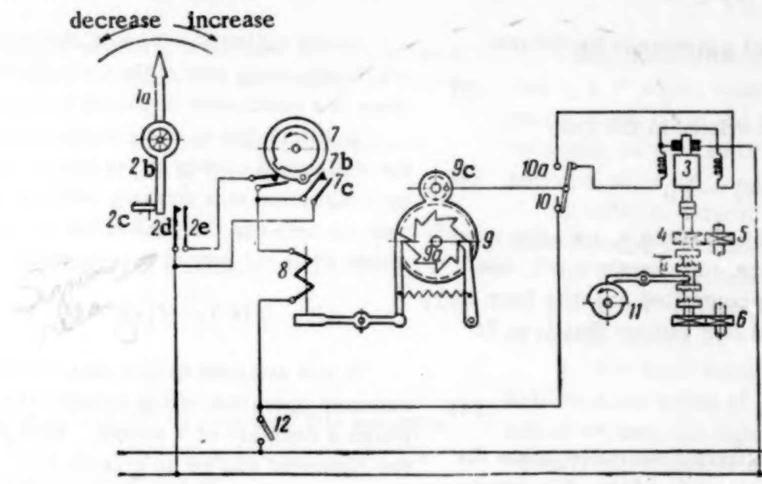
To provide stability of control, the variations of the controlled quantity during the establishment process are carried out nonmonotonically.

Basic to the scheme is a sp ecial device which provides selectivity in choosing an admissible disturbance and which transfers the motion from the class of inadmissible motions to that of admissible ones. This device is a relay which takes into account the direction (sign) of the velocity of the disturbed motion. It is because of this that we shall henceforth call it a signum-relay.

The signum-relay is made up of lever 2b (Fig. 1) friction-coupled to the axis of indicating instrument 1a which measures the controlled quantity p, so that the lever is dragged along by the pointer as p varies.

However, lever 2b can move only within the limits of a certain zone between contacts 2c and 2d. This will be called the insensitivity zone. As the pointer of the instrument moves to one side or the other, lever 2b will touch one of the contacts (2c or 2d) and will stop, while the pointer will continue its motion.

* This work was first published in the Trudy TsIAM in 1949; it is reprinted here without change.



In addition to the signum-relay, the following basic elements enter into the extremal controller scheme (Fig. 1):

1. The instrument which measures the controlled quan tity.

2. Reversible device 10 which is connected to the organ which acts on the controlling parameter.

3. A contact device (a commutator) which provides the periodic character of the artificial disturbance.

If the controlled quantity, whose extremum the controller seeks, depends on several parameters, then these latter are switched in by turn by means of device 4-5-6 in such manner that, during the control time for one of the parameters, the remaining ones either remain free, or are kept constant, or are varied in accordance with a given program.

Control is effected in the following manner. We assum that the controller is not switched in; then, the position of pointer 1a will not be changed (under the assumption that natural disturbances are absent), i.e., the parameter $\underline{v}$, acting on the quantity $\underline{p}$, does not change. With this, let it be assumed that the quantity $\underline{p}$ does not attain a maximum.

We now switch in knife switch 12. This starts the rotation of motor 3 (to one side or the other) and commutator disk 7.

With this, let the change of parameter $\underline{v}$ give rise to a decrease of $\underline{p}$. With this, instrument pointer 1a, in rotating counterclockwise, drags along lever 2b which, after traversing part of the insensitivity zone, closes contacts 2d and 2e which causes the operation of electromagnet 8 and stepping mechanism 9 which reverse motor 3 and change the direction of parameter $\underline{v}$. Thanks to this, the quantity $\underline{p}$ increases and, if the rotational speed of the disk of contact device 7 is sufficiently small, the quantity p attains an extremum and begins to move about a maximum.

To eliminate the possibility that the stimuli of disturbing forces would disrupt controller operation, contact device 7 (a commutator) was introduced.

Let the permissible deviation of parameter $\underline{v}$ from the value v_0 which corresponds to a maximum be some Δv. Then, the number of rotations of disk 7 is so chosen that, during one rotation, the change in the quantity $\underline{v}$ due to the action of motor 3 does not exceed Δv.

If, with such a choice for the number of rotations of disk 7, there now begins the action of disturbing forces, then two cases are possible:

a) The effect of the disturbance on either side is greater than the effect of variations in $\underline{v}$. In this case, changes in $\underline{v}$ from increasing $\underline{p}$ cease when $v_1 - \Delta v \leq v \leq v_1 + \Delta v$, where v_1 corresponds to the beginning of the disturbance.

b) The effect of the disturbance on either side is less than the effect of variations in $\underline{v}$. In this case, control proceeds normally until one reaches the value of $v^* + \Delta v$ (for disturbances which increase p) or $v^* - \Delta v$ (for the opposite case), where v^* is the quantity attainable in the absence of disturbing forces.

If the rotational speed of the commutators can not be considered small, and if a number of other design requirements do not hold, then the value of v^* attainable by the controller may differ from the optimal value v_0.

2. Investigation of the Establishment Process in the Extremal Controller

An extremal controller is a special nonlinear oscillatory system with one, two, or more degrees of freedom as a function of the character of the idealization.

With a number of far-reaching simplifications, in many cases the motion of the controlled system in the absence of external disturbances, i.e., for fixed values of v(in other words, with the extremal controller switched out), may be described by a linear differential equation

$$\frac{dp}{dt} = f(v) - p. \tag{1}$$

We shall assume that the function $f(v)$ is continuous in the interval $-\infty < v < +\infty$ and

$$f'(v_0) = 0, \quad f''(v) < 0.$$

We introduce the artificial disturbance by the law

$$v = kt. \tag{2}$$

Equation (1) may then be written in the form

$$\frac{dp}{dv} = \frac{1}{k}[f(v) - p]. \tag{3}$$

It is clear from (3) that, by choosing k, i.e., the velocity of the artificial disturbance, sufficiently small, one can make the deviation of the controlled quantity from its static value as small as desired. We assume that $k \ll 1$; with this, we may consider that

$$p = f(v). \tag{4}$$

With this condition, the extremal controller takes the form of a system with one degree of freedom. We denote by δ the magnitude of the signum-relay's insensitivity zone, and by Δt the commutator time (the size of the artificial disturbance's period).

Then, $\Delta v = k\Delta t$ is the quantity by which the argument v can change during the commutator time. We shall assume that, for the value $v = v_1$, the commutator operates and reversal occurs, after which the argument begins changing in one direction or the other.

We now show that the entire line of initial values is divided into a number of regions, where this decomposition depends on the relationship of δ and Δv, and also on the sign of k. The character of the control process is essentially different for each of these regions.

We shall later have occasion to use the following, almost self-evident lemma: if the continuous function $p=f(v)$ satisfies the condition $f'(v_0) = f''(v) < 0$ for $v \leq v_0$ then, for each pair of values v_a and v_b such that $v_a < v_b \leq v_0$, the following inequalities hold:

a) if we are given two numbers C_1 and C_2 $(0 < C_1 \leq C_2)$, then

$$f(v_b) - f(v_b - C_1) < f(v_a) - f(v_a - C_2), \tag{5}$$

b) if we are given numbers d_1 and d_2 such that

$$f(v_a) - f(v_a - d_1) = f(v_b) - f_1(v_b - d_2), \tag{6}$$

then

$$d_1 < d_2 \tag{6a}$$

and

$$v_a - d_1 < v_b - d_2. \tag{6b}$$

The proofs of these inequalities are given in the Appendix.

We shall consider the initial value $v \leq v_0$, assuming that reversal to a decreasing argument occurs for the value $v = v_1$. We assume that there exists a $v' < v_0$ such that

$$f(v') - f(v' - \Delta v) = \delta. \tag{A}$$

This can always be achieved by choosing a sufficiently small Δv or a sufficiently large δ.

We now show that, if (A) holds, the semiline of initial values $v_1 \leq v_0$ can be partitioned into the two regions:

$$v' < v_1 \leqslant v_0, \quad -\infty < v_1 < v'.$$

In the region $v' < v_1 \leq v_0$ the motion is stable, the amplitude being stationary for each initial value (i.e., there is a continuum of periodic motions).

In the region $v_1 < v'$, independently of which point of the region was chosen as the initial value, there occurs an approximation to a limiting periodic motion which is carried out between the boundaries of $v^* - \Delta v/2 \leq v \leq v^*$, where v^* is the root of the equation

$$f(v^*) - f\left(v^* - \frac{\Delta v}{2}\right) = \delta. \tag{B}$$

It was assumed earlier that, at the value $v = v_1$, there occurred operation of the commutator and reverser, after which a decrease of v ensued. With this, the decrease of the argument can be no greater than Δv: if the signum-relay operates, then the variation of the argument will be less than Δv, while if it does not operate the variation of the argument will equal Δv.

As a consequence of the decrease of the argument, the quantity p begins to decrease.

We now show that, if $v_1 < v'$ then, with a decrease of v by a certain definite amount a $(0 < a < \Delta v)$ given by (7) down to a value $v_2 = v_2(v_1)$, the signum-relay must operate; the controlled quantity will thereby decrease by the amount δ.

Indeed, the quantity a is defined by the condition that, with a decrease of the argument v by a, the insensitivity zone be traversed, i.e., that p be decreased by δ:

$$f(v_1) - f(v_1 - a) = \delta \tag{7}$$

or

$$f(v_1) - f(v_2) = \delta, \tag{7a}$$

where $v_2 = v_1 - a$.

We use the lemma at this point. By comparing (7) with (A), we note that (6a) holds if we take

$$v_1 = v_a, \quad v' = v_b, \quad a = d_1, \quad \Delta v - d_2;$$

consequently, $a < \Delta v$.

At the moment of operation of the signum-relay,

$$v = v_2 = v_1 - a.$$

Immediately after this, the argument begins to increase, continuing to do so until the next operation of the commutator.

The increase of the argument is by the quantity $\Delta v - a = \Delta v - (v_1 - v_2)$. At the moment of the next operation of the commutator, the argument has reached the value

$$v_3 = v_2 + [\Delta v - (v_1 - v_2)] = 2v_2 + \Delta v - v_1. \tag{8}$$

After this, the control cycle starts repeating.

Thus, as the result of one complete control cycle, the point v_1 on the axis of abscissas is translated to the point v_3 and, correspondingly, the point p_1 of the curve $p = f(v)$, with abscissa v_1, is translated to point p_3, with abscissa v_3.

It will be shown below that the image v_3 of point v_1 cannot leave the region $v < v'$.

Consequently, (8), where v_2 is defined by (7a), gives a transformation of the semiline $v_1 < v'$ into itself. We call this the transformation T.

The invariant points, which remain unmoved under this transformation, are defined by the condition

$$v_3 = v_1. \tag{9}$$

On the basis of (9) and (8) we obtain

$$v_2 = v_1 - \frac{\Delta v}{2} \tag{10}$$

or

$$a = \frac{\Delta v}{2}.$$

If we eliminate $\underline{a}$ from (7) by means of this last equation, and denote by v^* an invariant point, we obtain the following formula for determining v^*:

$$f(v^*) - f\left(v^* - \frac{\Delta v}{2}\right) = \delta. \tag{B}$$

3. <u>The Establishment Process in the Extremal Controller</u>

We now discuss the question as to the character of the establishment process, as well as the number of invariant points which satisfy Eq. (B).

After commutator operation, there begins the decrease of $\underline{v}$ down to the value of v_2, in accordance with the equation

$$v_2 = v_1 - a, \tag{8a}$$

where $\underline{a}$ is defined by (7).

With the reaching of the value $p = f(v_2)$, the signum-relay operates, and the argument begins increasing up to the value

$$v_3 = v_2 + (\Delta v - a) = v_1 - a + (\Delta v - a). \tag{8b}$$

It follows from (8b) that the value of v_3 will exceed v_1 if the difference Δv-a is greater than $\underline{a}$; for the entire range of values of v_1 in which this is the case, transformation T will translate every point of this region to the right.

If, in some region of values of v_1, the difference Δv-a is less than $\underline{a}$, then transformation T will here translate each point of this region to the left.

The condition $\Delta v - a > a$ leads to the inequality $\Delta v/2 > a$, while the condition $\Delta v - a < a$ leads to the inequality $\Delta v/2 < a$.

We consider the two cases $v_1 < v^*$ and $v^* < v_1 < v'$.

The quantity $\underline{a}$ is defined by (7). We compare it with (B) and note that the right members of (7) and (B) are identical. Consequently, in the case $v_1 < v^*$, the conditions of the lemma are met, if we take $v_1 = v_a$, $v^* = v_b$, $a = d_1$ and $\Delta v/2 = d_2$ but, in this case, we have, according to (6),

$$a < \frac{\Delta v}{2}. \tag{11a}$$

If we now consider the case when $v^* < v_1 < v'$, then the conditions of the lemma hold under the assumptions that

$$v^* = v_a, \quad v_1 = v_b, \quad a = d_2, \quad \frac{\Delta v}{2} = d_1. \tag{11b}$$

In this case,

$$a > \frac{\Delta v}{2}.$$

Thus, transformation T shifts all the points of the semiline $v_1 < v'$ which lie to the left of v^* to the right, and all points which lie to the right of v^* to the left. It thus follows that, on the semiline $v < v^*$ there is just the one immovable point v^*.

In order to determine the character of the establishment process, we must clear up still two more questions.

1. Will the distances ρ_i of the successive images v_3, v_5,... of point v_1 from fixed point v^* decrease, increase, or remain invariable ?

2. Are there conditions under which transformation T will translate points of the semiline $v < v'$ lying to the left of v^* into the region lying to the right of v^* and, conversely, can it translate points lying to the right of v^* into the region lying to the left of v^*?

We first prove that, no matter what point $v_1 < v^*$ we choose, its image v_3 will not leave the region $v < v'$.

Indeed, if the point lies to the right of v^* and to the left of v' then, as was proven earlier, transformation T translates it to the left; consequently, it can not move from the region $v^* < v_1 < v'$ into the region $v > v'$.

If, now, the point lies to the left of v^* then, by comparing (7) and (B), we see that, on the basis of (6b) of the lemma, $v_1 - a < v^* - \Delta v/2$; consequently,

$$v_3 = v_1 - a + (\Delta v - a) < v^* - \frac{\Delta v}{2} + (\Delta v - a) = v^* + \frac{\Delta v}{2} - a. \tag{8c}$$

But $a > 0$ and, therefore, by strengthening the inequality we get

$$v_3 < v^* + \frac{\Delta v}{2}. \tag{12}$$

We now compare (A) and (B). On the basis of (6b) of the lemma, we conclude that

$$v^* - \frac{\Delta v}{2} < v' - \Delta v \quad \text{or} \quad v^* + \frac{\Delta v}{2} < v'. \tag{13}$$

Thus, we always have $v_3 < v'$.

We now prove that, with successive applications of transformation T, the successive images v_3, v_5,... always approach closer to v^*. Let the distances of points $v_i (i = 1,2,...)$ from v^* be

$$\rho_i = v^* - v_i. \tag{14}$$

We form the function a':

$$a' = v^* - \frac{\Delta v}{2} - v_1 + a. \tag{15}$$

It is easily seen that

$$\rho_1 - a' = \frac{\Delta v}{2} - q. \tag{16}$$

To determine the character of the establishment process, it is necessary to consider the two cases: $\rho_1 > 0$ and $\rho_1 < 0$.

We first consider the case $\rho_1 > 0$. In this case $v_1 < v^*$. It was determined earlier that, for $v_1 < v^*$, $\Delta v/2 \geq a$.

Consequently, from (16),

$$\rho_1 - a' > 0 \quad (\rho_1 > 0).$$

Moreover, it was shown earlier that, if $v_1 < v^*$, then $v^* - \Delta v/2 > v - a$ and, consequently, $a' > 0$.

Therefore, a' can vary within the limits

$$\rho_1 > a' > 0 \tag{17a}$$

We now rewrite (8), taking (16) into account:

$$v_3 = v_1 + 2\left(\frac{\Delta v}{2} - a\right) = \tag{8d}$$

$$= v^* - \rho_1 + 2(\rho_1 - a') = v^* + \rho_1 - 2a'.$$

By bearing in mind that, on the basis of (14), for $i = 3$

$$v_3 = v^* - \rho_3, \tag{14a}$$

and by comparing (14a) and (8d), we obtain

$$\rho_3 = 2a' - \rho_1. \tag{C}$$

By taking (17a) into account, it is simple to conclude from (C) that ρ_3 can vary within the limits $-\rho < \rho_3 < \rho_1$. Whence

$$|\rho_3| < \rho_1 \quad (\rho_1 > 0). \tag{18a}$$

We turn now to the case when $\rho_1 < 0$, whereby $v' > v_1 > v^*$. In this case, $\Delta v/2 - a < 0$ and $v^* - \Delta v/2 < -v_1 - a$. Consequently, $a' < 0$ and $\rho_1 - a' < 0$.

It follows from this that a' can vary within the limits

$$\rho_1 < a' < 0. \tag{17b}$$

From whence, on the basis of (C), we conclude that

$$|\rho_3| < |\rho_1| \quad (\rho_1 < 0). \tag{18b}$$

It follows from (18a) and (18b) that, for any values $v_1 < v'$,

$$|\rho_3| < |\rho_1|. \tag{18}$$

It is thus shown that no matter where the point v_1 lies on the semiline $v < v'$, we always have $|\rho_3| < |\rho_1|$ i.e., repeated iteration of transformation T shifts its image ever closer to v^*. In other words, for any value of $v_1 < v'$, the extremal control process approximates to the limiting motion of period $\tau = \Delta v/k$, in which the argument changes periodically, with the same period τ, within the limits $v^* - \Delta v/2 \leq v \leq v^*$, and the corresponding controlled quantity p varies within the limits

$$f\left(v^* - \frac{\Delta v}{2}\right) \leqslant p \leqslant f(v^*).$$

Having (C), we can easily find the condition for which image v_3 of point v_1 turns out to be on the same side of the invariant point as v_1.

For this condition to be met, ρ_3 must keep the same sign as ρ_1.

We consider, as previously, the two cases: $\rho_1 > 0$ and $\rho_1 < 0$. Let $\rho_1 > 0$. Since, with this, we also have $a' > 0$ then, from (C), it follows that $\rho_3 > 0$ if

$$2a' > \rho_1. \tag{19a}$$

Now let $\rho_1 < 0$, for which $a' < 0$; consequently, $\rho_3 < 0$ if

$$|2a'| > |\rho_1|. \tag{19b}$$

Both conditions (19a) and (19b) reduce to the one:

$$\psi = |2a'| - |\rho_1| > 0. \tag{20a}$$

Thus, if the function $\psi > 0$, transformation T moves the image of point v_1 to the invariant point v^* while keeping it on the same side of v^*; if

$$\psi < 0, \tag{20b}$$

then the image of point v_1 is shifted by transformation T to v^* while simultaneously transferring over to the other side of v^*. If

$$\psi = 0 \tag{20c}$$

then image v_3 coincides with invariant point v^*, and the further successive images remain immovable.

Condition (20a) gives that property of the function $p = f(v)$ for which ρ_3 retains the sign of ρ_1. One may also give the boundaries for the point v_1 such that its image may not under any conditions coincide with v^*. These boundaries are defined by the condition that $\rho_3 \neq 0$.

Let $\rho_1 > 0$. We have from (16) that

$$a' = \rho_1 - \frac{\Delta v}{2} + a. \tag{16a}$$

On the basis of (C), the condition being sought reduces to the inequality $2a' > \rho_1$, i.e.,

$$2\left(\rho_1 - \frac{\Delta v}{2} + a\right) > \rho_1,$$

from whence

$$\rho_1 > \Delta v - 2a. \tag{21a}$$

But if $v_1 < v^*$, then a can vary within the limits $0 < a < \Delta v/2$. Therefore, if $\rho_1 > \Delta v$, then (21a) always holds. With this, $v_1 = v^* - \rho_1 = v^* - \Delta v$.

Now let $\rho_1 < 0$. With this, $v' > v_1 > v^*$ and the number a can vary within the limits

$$\frac{\Delta v}{2} < a < \Delta v.$$

The condition sought reduces to the inequality $2a' < \rho_1$ from whence, on the basis of (16a),

$$\rho_1 < \Delta v - 2a. \tag{21b}$$

Condition (21b) will be met if $\rho_1 < \Delta v - 2\Delta v = -\Delta v$. With this, $v_1 = v^* - \rho_1 > v^* + \Delta v$. Thus, if v_1 is at a distance from v^* whose modulus is greater than Δv, then the image of point v_1 cannot, under any conditions, coincide with v^*.

We now consider the case when $v_1 > v'$. As before, let a decrease of v commence after commutator operation; the decrease of v can continue either until the signum-relay operates or, failing that, until the commutator operates.

In the case under consideration ($v_1 > v'$) the signum-relay does not operate, since the controlled quantity p can change, with a decrease of the argument by Δv, only by the amount m, which is less than the insensitivity zone width δ.

This quickly follows from the lemma. Indeed, $m = f(v_1) - f(v_1 - \Delta v)$.

From a comparison with (A) we note that $v' < v_1 < v_0$ and the variations of the argument are identical; consequently, on the basis of part "a" of the lemma, $m < \delta$. Therefore, the reversal from an increase in v will occur, not by operation of the signum-relay, but by operation of the commutator, which will occur at the value $v_2 = v_1 - \Delta v$.

After commutator operation, the argument increases up to the value $v_3 = (v_1 - \Delta v) + \Delta v = v_1$. After this, there is again a reversal to decreasing v, operation of the signum-relay, but by operation of the commutator, since the signum-relay does not operate, reversal, etc.

Consequently, one full control cycle translates each initial value which lies in the region $v' < v < v_0$ into itself; there is a continuum of stationary amplitudes.

In other words, if the control process starts from a value of v lying in the region $v' < v < v_0$, then there will occur periodic oscillations of period $\tau = 2\ \Delta v/k$ for periodic variations of the argument within the limits $v_1 - \Delta v \geq v \geq v_1$ and a corresponding variation of the controlled quantity within the limits of $f(v_1 - \Delta v) \leq p \leq f(v_1)$.

Until now we have considered all the points of the semiline $v \leq v_0$ except the point $v = v'$, defined by (A).

All the values considered for $v = v_0$ correspond to definite actual oscillatory processes which occur in the extremal controller. This cannot be said of the initial value $v_1 = v'$.

In fact, according to (A), after a reversal to decreasing v occurs for the value v', the controlled quantity changes by δ for a change of argument of exactly Δv.

But with this there must simultaneously occur two actions: reversal due to signum-relay operation, and reversal due to commutator operation.

Such simultaneous operations have no real meaning. One might think that to this there would correspond a switching out of the entire reversing device; this is not admissible, despite the small probability that (A) holds. In practice, even if the initial value turns out to be equal to v', one must provide operation either of the commutator or of the signum-relay. Consequently, it would be possible to assign the value of v' to one of the regions.

We now consider controller operation if the initial value is $v > v_0$. If we assume that, after commutator operation initially, there occurs an increase of the argument then, by assuming that there exists a $v'' > v_0$ such that

$$f(v'') - f(v'' + \Delta v) = \delta,$$

we can transfer over all the results obtained for $v \leq v_0$ to the case when $v \geq v_0$. Instead of transformation T we shall have transformation T', given by the formula

$$v_3 = 2v_2 - \Delta v - v_1,$$

where v_2 is defined by the formula $f(v_1) - f(v_2) = \delta$.

Consequently, in this case one can partition the semiline of initial values $v \geq v_0$ into two regions: $v_0 \leq v_1 < v''$ and $v'' < v_1$. The value of v'' is defined as the root of (A').

In the region $v_0 \leq v_1 < v''$ the motion is stable, there being a continuum of periodic motions.

In the region $v'' < v_1$ there occurs an approximation to a limiting periodic motion, in which the argument varies periodically with period $\tau = \Delta v/k$ between the limits $v'^{*} \leq v \leq v'^{*} + \Delta v/2$, where v'^{*} is the root of the equation

$$f(v'') - f\left(v'' + \frac{\Delta v}{2}\right) = \delta.$$

In order to exhaust completely all the possibilities which can ensue for values of Δv and δ for which there exist $v = v'$ and $v = v''$, it is necessary to discard the supposition that, at the moment when the controller is switched in, the commutator necessarily is operated by a decrease of the controlled quantity p.

We note first of all, that if the initial value of v_1 for which the commutator operates on an increase in the controlled quantity is at a distance from v_0 equal to or greater than Δv, then this case leads to that already considered if one takes account, not of the value v_1, but of the value $v_1^! = v_1 \pm \Delta v$ ("+" for $v_1 < v_0$ and "-" for $v_1 > v_0$).

In fact, with this the following operation of the commutator will occur for a decrease of $p = f(v)$, which is a case already considered; the behavior of the controlled system will thus depend on whether $v_1^!$ lies in the region $v' < v_1 < v''$ or in the region $v_1 < v'$ and $v_1^! > v''$.

In the first case, there is a continuum of periodic motions; in the second case, there are either periodic motions about v^{*} and v'^{*} if, respectively, $v_1^! = v^{*}$ or $v_1^! = v'^{*}$, or an approximation to these limiting motions.

There still remains for consideration the case when the initial value of v_1 for which the commutator operates on an increase of the controlled quantity is separated from v_0 by a distance less than Δv. Let

$$v_1 = v_0 + b > v_0 \qquad (0 < b < \Delta v).$$

It is easily seen that, with a decrease of p from the maximum value $p = f(v_0)$, which will occur with decreasing v, down to the value $p = f(v_1 - \Delta v)$, the signum-relay does not operate, since the change in p, $\delta p = f(v_0) - f(v_0 - \Delta v)$ will be less than δ. Consequently, the following commutator operation occurs for the value:

$$v_2 = v_0 + b - \Delta v < v_0, \quad v_0 - v_2 = \Delta v - b < \Delta v.$$

After commutator operation, there begins an increase of $\underline{v}$ from the value v_2 to the value

$$v_3 = v_2 + \Delta v = v_0 + b - \Delta v + \Delta v = v_0 + b = v_1.$$

With the attainment of $v_3 = v_1$, there occurs a new operation of the commutator, and the cycle begins to repeat. Consequently, one full control cycle will transfer the initial value of v_1 into itself, i.e., there is a continuum of periodic motions. The case when $v_1 = v_0 - b$, $0 < b < \Delta v$, reduces to what has just been considered if one begins the investigation with the following operation of the commutator.

Thus, the partitioning of the line of initial values, $-\infty < v_1 < +\infty$, into regions depends on which side the argument begins to vary on at the moment when the controller is switched in.

Case A. If commutator operation gives rise to a decrease in $\underline{v}$, then the v_1 line is divided into three regions:

region L_1 $$v_1 < v'. \tag{22'}$$

region L_2 $$v' < v_1 < v'' + \Delta v. \tag{22''}$$

region L_3 $$v'' + \Delta v < v_1. \tag{22'''}$$

Case B. If commutator operation gives rise to an increase in $\underline{v}$, the regions are shifted to the left, and are the following:

region L_1' $$v_1 < v' - \Delta v, \tag{23'}$$

region L_2' $$v' - \Delta v < v_1 < v''. \tag{23''}$$

region L_3' $$v'' < v_1. \tag{23'''}$$

In the first and third regions there occur approximations to the limiting periodic motions while, in the second region, there is a continuum of periodic motions.

4. Effect of Variations of the Parameters Δv and δ

We have thus far assumed that Δv and δ have values such that there exist numbers $v' < v_0$ and $v'' > v_0$.

We now drop this limitation and consider the influence of changes in the parameters Δv and δ.

Let there exist real values of v' and v''. We shall fix $\Delta v = \Delta_1 v$ and we shall vary δ.

The decomposition into regions L_1, L_2, and L_3 depends on the character of the real roots of (A) and (A').

It is easily seen that, as δ_1 increases, the points v' and v'' move further away from v_0 while, for decreasing δ_1, the points v' and v'' move closer to v_0. For some definite values $\delta = \delta_1$ and $\delta = \delta_2$, the points v' and v'' coincide with v_0; the values of these are determined, respectively, from the expressions

$$f(v_0) - f(v_0 - \Delta v) = \delta_1. \tag{24}$$

$$f(v_0) - f(v_0 + \Delta v) = \delta_2.$$

Simultaneously with the change of position of the points v' and v'' on the line there is a change of position of the invariant points v^* and v^*_1, defined by (B) and (B'). These points move away from v_0 with increasing δ and approach v_0 for decreasing δ. Suppose that they coincide with v_0 for the values $\delta = \delta_3$ and $\delta = \delta_4$, defined by (25):

$$f(v_0) - f\left(v_0 - \frac{\Delta v}{2}\right) = \delta_3. \tag{25}$$

$$f(v_0) - f\left(v_0 + \frac{\Delta v}{2}\right) = \delta_4.$$

It is obvious that $\delta_3 < \delta_1$ and $\delta_2 < \delta_4$. We consider the case when the curve $p = f(v)$ is symmetric about the line $v = v_0$. Then, $\delta_1 = \delta_2$ and $\delta_3 = \delta_4$. Let $\delta_3 < \delta < \delta_1$. Consequently, there exists no v' which satisfies (A). Let $f(v_0) - f(v_0 - a_1) = \delta$ and consequently, $a_1 < \Delta v$. Therefore, operation of the signum-relay occurs to the left of v_0 if the initial value lies to the right of v_0 in the region L_1 [Condition (26)].

With this, the regions of (22) are deformed, and assume the forms:

region L_1 $$v_1 < v_0 + \Delta v - a. \tag{26'}$$

region L_2 $$v_0 + \Delta v - a_1 < v_1 < v_0 + a_1. \tag{26''}$$

region L_3 $$v_0 + a_1 < v_1. \tag{26'''}$$

We shall continue to decrease δ. If $\delta = \delta_3$, i.e., if v^* coincides with v_0, then $a_1 = \Delta v/2$; then,

$$v_0 + \Delta v - a_1 = v_0 + \frac{\Delta v}{2}, \quad v_0 + a_1 = v_0 + \frac{\Delta v}{2}.$$

Consequently, region L_2 shrinks to a point. There remain two regions. Region L_1 : $v_1 < v_0 + \Delta v/2$; region L_3: $v_0 + \Delta v/2 < v_1$.

With still greater decrease of δ, the motion loses its simple character and becomes a multiple motion.

It may be shown that the mode of multiple oscillations is not effective in practice here, since oscillations are possible with this that have greater amplitudes than the simple oscillations.

It is therefore meaningful to decrease δ and to increase Δv until v^* gets close to v_0, but not beyond that; this is the more advantageous since, generally speaking, it is quite difficult to obtain a small δ, but a decrease in Δv is very desirable from the point of view of increasing the stability of the control mode with respect to external disturbances.

We have been changing δ with Δv held fixed. It is easily seen that an increase in Δv acts analogously to a decrease in δ.

We can consider the case of asymmetry analogously.

5. Consideration of a Concrete Problem

Let the function $p = f(v)$ have the form

$$p = -v^2. \tag{27}$$

We shall consider initially a value of $v_1 > v_0 = 0$. If Δv and δ are given, then the magnitude of v^* is defined by the equation

$$f(v^*) - f\left(v^* - \frac{\Delta v}{2}\right) = \delta; \tag{28}$$

from which, based on (27),

$$v^* = \frac{\Delta v}{4} - \frac{\delta}{\Delta v} \qquad (v^* \leqslant 0).$$

Equation (28) is meaningful when $\delta \geq \Delta v^2/4$. Region (A) is isolated by the boundaries v' and $v_0 = 0$, where v', in accordance with (A), is defined by the formula

$$v' = \frac{\Delta v}{2} - \frac{\delta}{2\Delta v} \qquad (v' \leqslant 0). \tag{29}$$

We now determine the character of the establishment process in accordance with (20a) and (20b). The character of the process we are seeking depends on the sign of the expression $\psi = |2a'| - |\rho_1|$. We now construct this function. According to (14),

$$a' = \left(v^* - \frac{\Delta v}{2}\right) - (\delta_1 - a),$$

where $\underline{a}$ is the root of (7).

From (7) we obtain the quadratic equation in $\underline{a}$:

$$a^2 - 2v_1 a - \delta = 0;$$

from whence

$$a = v_1 + \sqrt{v_1^2 + \delta}.$$

(The plus sign is chosen so that $a > 0$.)

Consequently,

$$a' = \left(\frac{\Delta v}{4} - \frac{\delta}{\Delta v} - \frac{\Delta v}{2}\right) - \left[v_1 - \left(v_1 + \sqrt{v_1^2 + \delta}\right)\right]$$

Further,

$$\rho_1 = v^* - v_1 = \left(\frac{\Delta v}{4} - \frac{\delta}{\Delta v} - v_1\right).$$

Inequality (20a) takes the form

$$\psi = \pm\left[-\frac{3}{4}\Delta v - \frac{\delta}{\Delta v} + v_1 + 2\sqrt{v_1^2 + \delta}\right] > 0.$$

(The plus sign is taken for $v_1 < v^*$, the minus sign for $v_1 > v^*$; this expression is meaningful for $\delta \geq \Delta v^2/4$.

It is easily shown that this expression cannot be satisfied for any arbitrary v_1. It follows from this that it is impossible to choose values of Δv and δ such that the approximation to v^* occurs for any v_1 on one side only. However, for certain regions of values of Δv and δ, one can cite such a point $v_1 = v_u \neq v^*$ whose image v_3 coincides with the invariant point v^*.

In fact, it follows from (20c) that this point is defined by the condition that $\psi = 0$.

Whence,

$$v_1 = \frac{-\left(\frac{3}{2}\Delta v + \frac{2\delta}{\Delta v}\right) \pm \left(\frac{4\delta}{\Delta v} - 3\Delta v\right)}{6}.$$

The minus sign defines the invariant point v^*. The plus sign gives the point v_u sought:

$$v_u = \frac{3}{4}\Delta v + \frac{1}{3}\frac{\delta}{\Delta v}. \tag{30}$$

Formula (30) gives a negative value of v_u for $\delta < 2.25\Delta v^2$, while it follows from (28) that v^* has a minus sign [the condition for applicability of (28)] if $4\delta > \Delta v^2$.

Consequently, the values of v_u are meaningful in the region

$$0.25\Delta v^2 < \delta < 2.25\Delta v^2. \tag{31}$$

We now consider the entire v_1 line, and construct the regions of initial values given by (23) and (26). So long as there exist values $v' < v_0 = 0$, i.e., so long as $\delta > \Delta v^2$ [this follows from (29)], the line of initial values is partitioned into the following three regions (22):

1) region L_1

$$v_1 < \frac{\Delta v}{2} - \frac{\delta}{2\Delta v};$$

2) region L_2

$$\frac{\Delta v}{2} - \frac{\delta}{2\Delta v} < v_1 < \frac{\delta}{2\Delta v} - \frac{\Delta v}{2} + \Delta v = \frac{\delta}{2\Delta v} + \frac{\Delta v}{2};$$

3) region L_3 $\qquad \dfrac{\delta}{2\Delta v} + \dfrac{\Delta v}{2} < v_1.$

If δ is included in the interval $0.25\,\Delta v^2 < \delta < \Delta v^2$, then the line of initial values is partitioned into the regions of (26) [where the quantity a_1 is defined by the condition $f(v_0) - f(v_0 - a_1) = \delta$, but $v_0 = 0$, and, consequently, $a_1 = \sqrt{\delta}$]:

1) region L_1

$$v_1 < v_0 + \Delta v - \sqrt{\delta};$$

2) region L_2

$$v_0 + \Delta v - \sqrt{\delta} < v_1 < v_0 + \sqrt{\delta};$$

3) region L_3

$$v_0 + \sqrt{\delta} < v_1.$$

As δ decreases from $\delta = \delta_1 = \Delta v^2$ to $\delta = 0.25\Delta v^2$, region L_2 decreases and, for $\delta = 0.25\,\Delta v^2$, shrinks to a point. With further decreases in δ, the motion becomes multiple.

Returning to (31), we note that, in the region of values $\delta > 2.25\,\Delta v^2$, there occurs an asymptotic approach to periodic motions (if v_1 is chosen in regions L_1 or L_3); in the region of (31) the establishment process can be either asymptotic or finite; in the region of $\delta < 0.25\Delta v^2$, there are multiple oscillations.

The character of the possible motions in the system under consideration depends on the two essential parameters Δv and δ.

Figure 2 shows the Δv, δ parameter plane.

In the region bounded by the arc of the parabola $\Delta v^2 = \delta$ and the positive δ semiaxis, there is a rigid excitation mode with a decomposition of the initial values in accordance with the domains of (22).

In the region between the arcs of the parabolas $\Delta v^2 = 4\delta$ and $\Delta v^2 = \delta$, there are rigid modes with decomposition in accordance with the regions of (26).

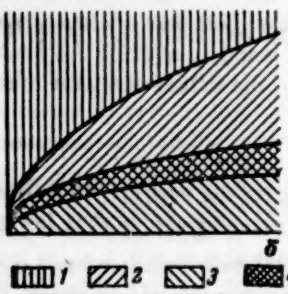


Fig. 2. 1 is the multiple oscillatory mode, 2 is the rigid mode, with decomposition into the regions of (26), 3 is the asymptotic approach to a periodic motion (the decomposition into the regions of (22)), 4 is the finite and asymptotic establishment processes (decomposition into the regions of (22)).

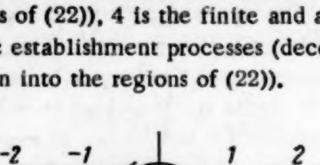


Fig. 3.

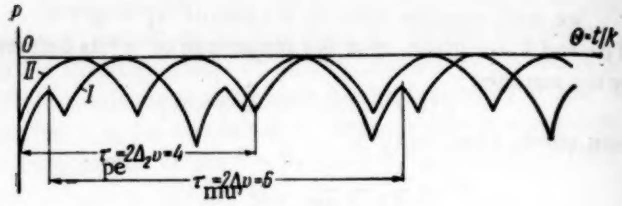


Fig. 4.

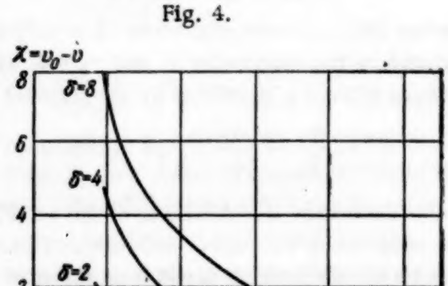
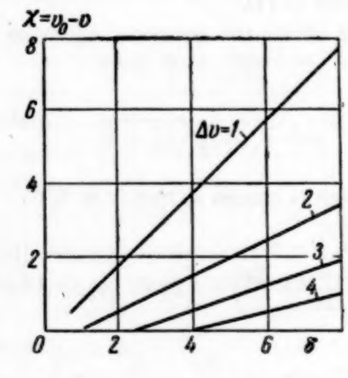


Fig. 5



The asymptotic approximation to periodic motion occurs in the region bounded by the arc of the parabola $9\Delta v^2 = 4\delta$ and the positive δ semiaxis.

The region $\Delta v^2 > 4\delta$ is characterized by the presence of a multiple oscillatory mode.

Figure 3 shows the phase diagram of the multiple periodic oscillations for $\Delta v = 3$ and $\delta = 1$.

Figure 4 shows the development of the multiple periodic oscillations in time (curve I), where the axis of abscissas is laid out with time in the scale of $1/k$. This same figure gives the development (curve II) in time of the simple periodic oscillations for the limiting value of $\Delta v = 2$ for the chosen value of $\delta = 1$. It is clear, from a comparison of curves I and II, that the multiple oscillations are less suitable, since their amplitudes are greater than those of the simple oscillations.

In conclusion, we consider the question of the influence of the system's parameters on the inaccuracy of the extremal control operation. By "inaccuracy of operation" we shall understand the difference between the value v_0 of the argument corresponding to the true extremal value of the controlled quantity and the limiting value v^* which can be attained as a result of the extremal controller's action.

We denote this difference by χ:

$$\chi = v_0 - v^* = \frac{\delta}{\Delta v} - \frac{\Delta v}{4}. \tag{32}$$

Expression (32) is meaningful so long as $\chi \geq 0$, since the case $\chi < 0$ corresponds to the case when the motion becomes multiple, and when (28) can therefore not be applied.

Figure 5 gives graphs of the quantity χ as a function of Δv for different fixed values of δ. It is clear from this figure that each value of δ has its corresponding value of $\overline{\Delta v}$ for which the inaccuracy χ vanishes. With decreasing Δv the inaccuracy increases, this increase not being very large for values of Δv close to $\overline{\Delta v}$; for significant decreases in Δv the magnitude of χ begins to increase more rapidly since, in (32), the first, hyperbolic-type, term begins to play a predominant role.

Figure 6 gives graphs of the quantity χ as a function of δ for different fixed values of Δv. This function is given by a family of straight lines with slopes greater than zero.

A comparison of Figs. 5 and 6 shows that the increases in δ have a lesser effect on the increase of the inaccuracy than do decreases in Δv.

Appendix

Lemma. If the continuous function $p = f(v)$ satisfies the conditions $f'(v_0) = 0$, $f''(v) < 0$ for $v \leq v_0$ then, for each pair of values v_a and v_b ($v_a < v_b \leq v_0$), the following inequalities hold:

a) if there are given two numbers c_1 and c_2 ($0 < c_1 \leq c_2$), then

$$f(v_b) - f(v_b - c_1) < f(v_a) - f(v_a - c_2); \qquad (5)$$

b) if there are given two numbers d_1 and d_2 such that

$$f(v_a) - f(v_a - d_1) = f(v_b) - f(v_b - d_2), \qquad (6)$$

then

$$d_1 < d_2 \qquad (6a)$$

and

$$v_a - d_1 < v_b - d_2. \qquad (6b)$$

Proof. We first note that, since $f''(v) < 0$ then, for decreasing v_1, the function $f'(v)$ increases monotonically.

First, we prove (5).

a) We assume initially that $v_a < v_b - c_1$; in this case, the intervals $v_a - c_2 \leq v \leq v_a$ and $v_b - c_1 \leq v \leq v_b$ are non-intersecting.

With this,

$$[f'(v)]_{v<v_a} > [f'(v)]_{v>v_b-c}. \qquad (33)$$

Further,

$$f(v_b) - f(v_b - c_1) = \Delta P_1 = \int_{v_b-c_1}^{v_b} f'(v)\,dv, \qquad (34a)$$

$$f(v_a) - f(v_a - c_2) = \Delta P_2 = \int_{v_a-c_2}^{v_a} f'(v)\,dv. \qquad (34b)$$

On the basis of (33), and bearing in mind that $c_1 \leq c_2$, we conclude that $\Delta P_1 < \Delta P_2$.

b) We now assume that $v_a > v_b - c_2$. In this case, the intervals $v_a - c_2 \leq v \leq v_a$ and $v_b - c_1 \leq v \leq v_b$ partially intersect each other; with this, the segment $v_b - c_1 \leq v \leq v_a$ is common to both intervals. Consequently, on this segment, the changes in the function $f(v)$ will also be identical.

It remains to compare the changes of the function $f(v)$ on the intervals $v_a - c_2 \leq v \leq v_b - c_1$ and $v_a \leq v \leq v_b$. But, for these intervals, we have case "a". Consequently, here also $\Delta P_1 < \Delta P_2$. Thus, (5) is proven.

Now we prove (6a).

We fix the quantity c_1 in the left member of (5), introducing the notation $c_1 = d_2$. We now, in the right member of (5), change c_2 (denoting it by d_1) in such fashion that the inequality becomes an equality.

Since, in (34b), $f'(v)$ retained its sign over the entire interval of integration, the quantity defined by the integral in (34b) can be decreased only by decreasing the length of this interval, whereby, on the basis of (33), equality can be reached only for $d_1 < d_2$.

As for (6b), it is obvious.

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ELECTROMAGNETIC CONTROL-ELEMENT DYNAMICS

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The equation of motion is derived for a rotary type electromagnetic control element, and a comparison is made with experimental results.

In automatic control and regulation systems, extensive use has been made of electric controlling elements in which electrical quantities are transformed to mechanical ones. Such elements (electromagnetic and electrodynamic) are used in remote control systems, in electrohydraulic servomechanisms and in other devices of automation technology.

Despite the widespread usage of electric controlling elements, their dynamics are still insufficiently explicated. The existing works [1] in this area treat only individual questions, and lack a general character.

In the present paper we consider the dynamics of an electromagnetic polarized controlling element, since it is the most widely used type.

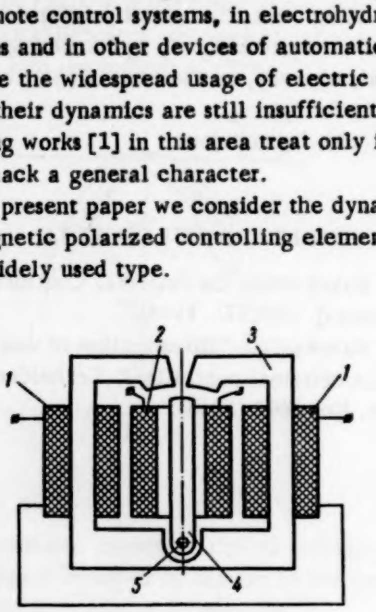


Fig. 1

A schematic drawing of the electromagnetic control element is shown in Fig. 1, where 1 is the excitation coil (superposed magnetization), 2 is the control coil, 3 is the magnetic circuit, 4 is the armature, and 5 is the output shaft. When a controlling signal is applied to coil 2, output shaft 5 is rotated by a definite angle.

We now write the general form of the equation of motion for the electromagnetic control element's armature:

$$J\ddot{\theta} = M_{d} - M_{i}. \qquad (1)$$

Here, J is the armature's moment of inertia, θ is the armature's angle of rotation, M_d is the driving torque, and M_i is the impedance torque of the armature's motion.

The driving torque, equal to

$$M_{d} = PR \qquad (2)$$

(where P is the force engendering the armature rotation and R is the radius of application of the force), depends on the current flowing in the control coil.

The impedance torque is made up of the load torque M_L, the friction torque in the bearings M_F and the damping torque M_v.

In considering the operation of an electromagnetic control element in any actual control scheme, it is necessary to take into account both the form of the load and the frictional force in the bearings arising when an external force is applied to the armature axis [2]. For investigating the dynamics of an inherently electromagnetic control element removed from any control scheme (and these may be very diverse), there is the possibility of not taking the load on the control element's armature into account. In this case, the impedance torque will be determined by the friction in the bearings under the action of the armature's own weight, and by the damping which arises when the armature moves within the surrounding air, i.e.,

$$M_{i} = Qrf \operatorname{sign} \dot{\theta} + kq\dot{\theta}, \qquad (3)$$

where Q is the armature's weight, r is the radius of the armature shaft, f is the coefficient of friction in the bearings, k is a proportionality factor, and q is the coefficient of viscous friction.

By substituting (2) and (3) in (1), we get

$$J\ddot{\theta} = PR - Qrf \operatorname{sign} \dot{\theta} - kq\dot{\theta}. \qquad (4)$$

The work expended in moving the control element's armature, as in every polarized electromagnetic system, is carried out at the expense of two energy sources: the controlling (working) and the polarizing sources. In the general case, the force acting on the armature of the system under consideration can be presented in the form of a sum of three forces [3]

$$P = P_1 + P_2 + P_3,$$

where P_1 is the component due to the polarizing source (or sources), which exists whether or not there is current in the control winding, P_2 is the component due to the controlling (working) source (or sources), P_3 is the component arising from the interaction of the magnetic fluxes of the controlling and polarizing sources which actually determines the direction of armature motion.

For the case when the polarizing and working sources are coils, these components equal

$$P_1 = \Sigma \frac{(i_e w_e)^2}{2} \frac{dG_e}{dy}, \quad P_2 = \Sigma \frac{(i_c w_c)^2}{2} \frac{dG_c}{dy},$$

$$P_3 = \Sigma \frac{i_e w_e i_c w_c}{2} \frac{dG_{ec}}{dy},$$

where i_e and i_c are the currents flowing in the excitation and control coils, respectively, w_e and w_c are the number of turns in the excitation and control coils, G_e is the circuit's conductance with respect to the magnetic flux of the excitation coil, G_c is the circuit conductance with respect to the control coil's magnetic flux, G_{ec} is the circuit conductance determining the magnitude of the portion of the excitation coil's magnetic flux in the control winding, y is the linear deviation of the armature from the neutral position*.

As is obvious from these expressions, only the sign of the third component is determined by the polarity of the current in the control winding. The summation signs in all these expressions are indicative of the fact that polarized systems, as a rule, have complicated configurations of magnetic circuits with several controlling and polarizing sources.

The system under consideration is a differential one. In it (Fig. 1) there are two excitation coils on the arms of the magnetic circuit, and one control coil on the cross piece. With no controlling signal present, the electromagnet's armature is acted upon by the forces generated by the first and second excitation coils. In this case, the component P_1 is the sum of these two forces, and equals

$$P_1 = \frac{(i_1w_1)^2}{2}\frac{dG_1}{dy} + \frac{(i_2w_2)^2}{2}\frac{dG_2}{dy}, \qquad (5)$$

where G_1 and G_2 are the circuit's conductances with respect to the magnetic fluxes of the first and second excitation coils, respectively, i_1, i_2 and w_1, w_2 are the currents and numbers of turns in the first and second excitation coils, respectively.

Since the coils are identical ($w_1 = w_2 = w_e$) and are connected in series ($i_1 = i_2 = i_e$), the expression may be rewritten in the form

$$P_1 = \frac{(i_e w_e)^2}{2}\left(\frac{dG_1}{dy} + \frac{dG_2}{dy}\right). \qquad (6)$$

The second component P_2, generated by the control coil, can be expressed by the formula

$$P_2 = \frac{(iw)^2}{2}\frac{dG}{dy}, \qquad (7)$$

where i and w are the current and number of turns in the control coil, G is the circuit conductance with respect to the magnetic flux of the control coil.

The third component, created by the interaction of the two counter-directed magnetic fluxes of the first and second excitation coils with the magnetic flux of the control coil, equals

$$P_3 = \frac{iwi_e w_e}{2}\left(\frac{dG_{2c}}{dy} - \frac{dG_{1c}}{dy}\right), \qquad (8)$$

where G_{1c} and G_{2c} are the conductances which determine the magnitude of the magnetic fluxes in the control coil which correspond to the first and second excitation coils.

In order to express P_1, P_2, and P_3 in terms of the controlling element's parameters, we use its replacement circuit (Fig. 2). Figures 3 and 4 show the nomenclature of the basic design parameters of the control element. With this nomenclature, we can write that the impedances of the left and right air gaps equal, respectively,

$$R_{P1} = \frac{\delta}{(c-y)d}, \qquad R_{P2} = \frac{\delta}{(c+y)d}, \qquad (9)$$

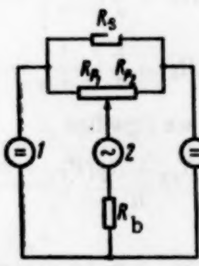


Fig. 2. 1 is the excitation winding's mmf, 2 is the control winding's mmf.

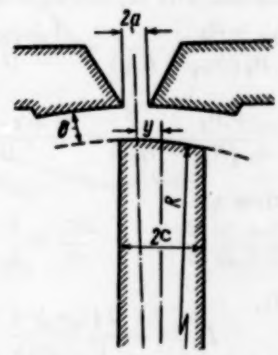


Fig. 3. Basic design parameters of the electromagnetic control element.

Fig. 4. Design parameters of the armature's bearing portion.

and that the air gap between the bearing and the armature equals

$$R_b = \frac{\delta}{ld}, \qquad (10)$$

where $c = b - a$, $l = hk_1$, $k_1 = \delta/\Delta$, d is the armature thickness.

We take $R_s = \infty$, and calculate a number of auxiliary quantities:

$$R_{P1} + R_{P2} = \frac{2c\delta}{(c^2 - y^2)d}.$$

$$R_{P1} - R_{P2} = \frac{2y\delta}{(c^2 - y^2)d}.$$

*We consider small angular deviations of the armature, within the limits of ± 1 degree.

$$R_{P1}R_{P2} = \frac{\delta^2}{(c^2-y^2)\,d^2}.$$

$$R_{P1}R_{P2} + R_\delta(R_{P1}+R_{P2}) = \frac{\delta^2}{(c^2-y^2)\,d^2}\left(1+\frac{2c}{l}\right).$$

$$R_{P1}+R_\delta = \frac{\delta}{d}\left(\frac{1}{c-y}+\frac{1}{l}\right).$$

$$R_{P2}+R_\delta = \frac{\delta}{d}\left(\frac{1}{c+y}+\frac{1}{l}\right).$$

To determine P_1 we first find

$$R_1 = \frac{1}{G_1} = \frac{R_{P1}R_{P2}+R_\delta(R_{P1}+R_{P2})}{R_{P2}+R_\delta}. \tag{11}$$

$$R_2 = \frac{1}{G_2} = \frac{R_{P1}R_{P2}+R_\delta(R_{P1}+R_{P2})}{R_{P1}+R_\delta}.$$

The conductances will be, correspondingly,

$$G_1 = \frac{R_{P2}+R_\delta}{R_{P1}R_{P2}+R_\delta(R_{P1}+R_{P2})} = \frac{(l+c+y)(c-y)\,d}{(l+2c)\,\delta}. \tag{12}$$

$$G_2 = \frac{R_{P1}+R_\delta}{R_{P1}R_{P2}+R_\delta(R_{P1}+R_{P2})} = \frac{(l+c-y)(c+y)\,d}{(l+2c)\,\delta}.$$

and their derivatives are

$$\frac{dG_1}{dy} = -\frac{l+2y}{l+2c}\,\frac{d}{\delta}. \qquad \frac{dG_2}{dy} = \frac{l-2y}{l+2c}\,\frac{d}{\delta}. \tag{13}$$

Consequently,

$$P_1 = -\frac{2\,(i_e w_e)^2\,d}{(l+2c)\,\delta}\,y. \tag{14}$$

The minus sign shows that component P_1 is directed opposite to the translation y.

Component $P_2 = 0$ by virtue of the fact that the conductance is constant, since

$$R_0 = \frac{1}{G} = R_\delta + \frac{R_{P1}R_{P2}}{R_{P1}+R_{P2}} = R_\delta + \frac{\delta}{2cd} \tag{15}$$

and its derivative dG/dy equals zero.

To determine component P_3 we should find the expressions for G_{1c} and G_{2c}. For this, we sketch the replacement circuit (Fig. 5) and determine which portion of the fluxes engendered by each excitation coil will flow through the control coil. The action of each excitation coil will be considered separately. Based on Ohm's laws and Kirchhoff's laws, we get

$$G_{1c} = \frac{R_{P2}}{R_{P1}R_{P2}+R_\delta(R_{P1}+R_{P2})},$$

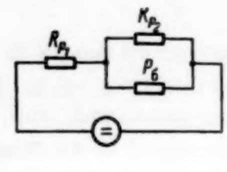
$$G_{2c} = \frac{R_{P1}}{R_{P1}R_{P2}+R_\delta(R_{P1}+R_{P2})}. \tag{16}$$

Consequently,

$$G_{2c}-G_{1c} = \frac{R_{P1}-R_{P2}}{R_{P1}R_{P2}+R_\delta(R_{P1}+R_{P2})}. \tag{17}$$

After substituting of the impedance values, we get

$$G_{2c}-G_{1c} = \frac{2dly}{\delta\,(l+2c)}, \tag{18}$$



whence

$$\frac{d\,(G_{2c}-G_{1c})}{dy} = \frac{2ld}{\delta\,(l+2c)}. \tag{19}$$

Consequently,

$$P_3 = iwi_e w_e\,\frac{ld}{\delta\,(l+2c)}. \tag{20}$$

Thus, the general expression for the force will have the form

$$P = P_3 + P_1 \tag{21}$$

or

$$P = iwi_e w_e\,\frac{ld}{\delta\,(l+2c)} - \frac{2\,(i_e w_e)^2\,d}{\delta\,(l+2c)}\,y. \tag{22}$$

As is clear from this expression, the component P_1 of the force on the armature acts as a spring, rotating the armature after cessation of the action of the controlling signal to the original position. It may therefore be called an electrical spring.

Figure 6 shows the family of static characteristics $M = f(\theta)$ for various currents i in the control winding, these curves having been obtained experimentally. These graphs substantiate the correctness of our computations.

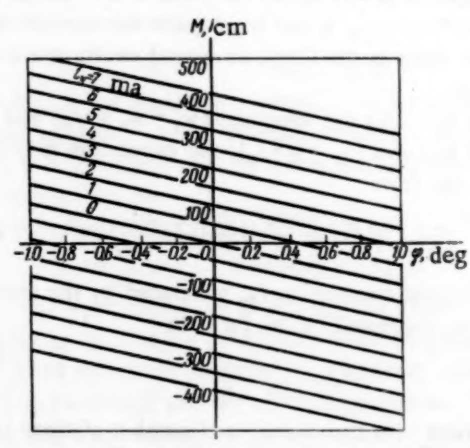


Fig. 6. Static characteristics of the electromagnetic control element.

For small angles of rotor deviation, we may assume that

$$y = \theta R. \tag{23}$$

Then,

$$P = \frac{iwi_e w_e ld}{\delta\,(l+2c)} - \frac{2\,(i_e w_e)^2\,Rd}{\delta\,(l+2c)}\,\theta \tag{24}$$

or

$$P = Ai - B\theta, \tag{25}$$

where

$$A = \frac{wi_e w_e ld}{\delta\,(l+2c)}, \quad B = \frac{2\,(i_e w_e)^2\,Rd}{\delta\,(l+2c)}$$

By substituting the expression obtained for the force P in (4), we obtain

$$J\ddot{\theta} = (Ai - B\theta)\,R - kq\dot{\theta} - Qrf\,\mathrm{sign}\,\dot{\theta}$$

or

$$\frac{J}{AR}\ddot{\theta} + \frac{kq}{AR}\dot{\theta} + \frac{B}{A}\theta + \frac{Qfr}{AR}\,\mathrm{sign}\,\dot{\theta} = i.$$

By going over to dimensionless symbols, we get the equation describing the armature motion in the following form:

$$m\ddot{\varphi} + \gamma\dot{\varphi} + \beta\varphi + \alpha\,\mathrm{sign}\,\dot{\varphi} = I, \qquad (26)$$

where

$$m = \frac{J\delta\,(l + 2c)\,\theta_m}{i_e w_e dlwRi_m}, \qquad \gamma = \frac{kq\delta\,(l + 2c)\,\theta_m}{i_e w_e dlwRi_m}, \qquad \beta = \frac{2i_e w_e R\theta_m}{lwi_m},$$

$$\alpha = \frac{\theta rf\delta\,(l + 2c)\,\theta_m}{i_e w_e dlwRi_m}, \qquad \varphi = \frac{\theta}{\theta_m}, \qquad I = \frac{i}{i_m},$$

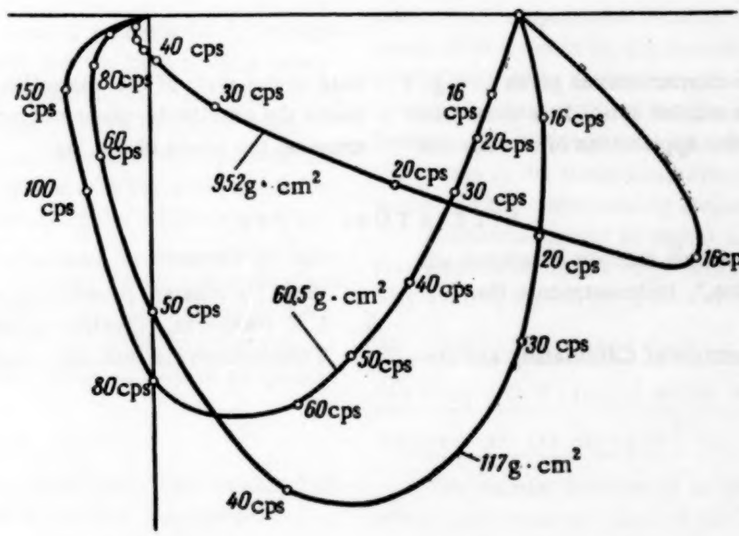


Fig. 7. Experimental amplitude-phase characteristics.

θ_m and i_m are the basic magnitudes of the angle of deviation and the control current.

In (26), the term α sign $\dot{\varphi}$, characterizing the friction in the bearings, is a negligibly small quantity, since the armature axis is supported in ball bearings and operates with flood lubrication. In view of this, the motion of the controlling device's armature can be described by an equation of the form

$$T_2^2\ddot{\varphi} + T_1\dot{\varphi} + \varphi = \frac{1}{\beta}I. \qquad (27)$$

where

$$T_2^2 = m/\beta, \quad T_1 = \gamma/\beta.$$

Expression (27) is the usual form for writing the equation of a second-order oscillatory link. Its transfer function can be given in the following form:

$$W(p) = \frac{1}{\beta\,(T_2^2p^2 + T_1p + 1)}. \qquad (28)$$

The form of the frequency characteristic of such a link is widely known, but the data which characterize the frequency characteristic of an actual electromagnetic controlling device are of definite interest.

The amplitude and phase frequency characteristics of a type RÉP electromagnetic controlling device were recorded for a controlling current of i = 2 ma and for different inertial loads on the armature axis (for different moments of inertia)†. The armature's inherent moment of inertia J equalled 60.5 g · cm². The table presents the data obtained for a control current of i = 2 ma; the basic magnitude of armature angular rotation was taken to be $\theta_m = 0.4°$. The basic value of control current was taken to be $i_m = 2$ ma.

The quantity N in the table is the ratio of the dimensionless amplitude of armature angle of rotation to the dimensionless control current, i.e.,

$$N = \frac{\theta_a}{\theta_m} \Big/ \frac{i}{i_m},$$

and the quantity ψ is the lag phase.

† The experiments were carried out by L. P. Levin, G. Yu. Chubarova, and G. M. Val'kova.

Frequency, cps	Moment of inertia g · cm^2								
	60.5			117			952		
	θ_a, deg	N	ψ, deg	θ_a, deg	N	ψ, deg	θ_a, deg	N	ψ, deg.
16	0.376	0.94	13	0.44	1.11	15	0.64	1.6	24
20	0.372	0.93	20	0.48	1.2	30	0.32	0.8	35
30	0.376	0.94	30	0.53	1.33	42	0.124	0.31	55
40	0.40	1.0	46	0.53	1.33	75	0.05	0.126	85
50	0.42	1.06	60	0.32	0.8	90	0.04	0.1	108
60	0.44	1.11	70	0.16	0.4	110	0.028	0.07	120
80	0.396	0.99	90	0.08	0.2	140	0.02	0.05	150
100	0.20	0.5	110	0.05	0.126	160	0.014	0.035	180
150	0.12	0.3	140	0.025	0.063				

The amplitude-phase characteristics given in Fig. 7 were constructed from the tabular data. It is clear from these characteristics that the application of an external load to the shaft of the controlling element's armature deforms the amplitude-phase characteristic by way of increasing the phase of the lag.

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DIFFERENTIATION OF SLOWLY VARYING SIGNALS

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The paper considers methods of differentiating slowly varying electrical signals with respect to time, taking into account the nonlinearities inherent in actual devices, as well as methods of replacing the differentiation operation by a determination of the increments of the input quantity during fixed intervals of time.

Posing of the Problem

In the present work we consider methods for differentiating slowly varying signals with spectra from zero up to several cps. In such a frequency spectrum the signal derivative will be small, and the receiving devices will have finite sensitivities. Therefore, the differentiator must provide amplification of the signal derivative. On the other hand, for slowly varying signals, the inertia of the differentiator itself has a lower value.

It is convenient to open the investigation of possibilities of differentiating slowly varying signals by comparing the frequency characteristics of the corresponding differentiating device with the frequency characteristics of an ideal differentiator

$$\Phi_{id}(j\Omega) = T_{d.id}\, j\Omega,$$

where $T_{d.id}$ is the derivative coefficient (time constant) of the ideal differentiator and Ω is circular frequency.

On the basis of such a comparison one can establish, for the real differentiator, a working range of frequency characteristic in which its distortion will not exceed some established quantity. We adopt the convention in this work that the difference of the phase frequency characteristics of the ideal and the real differentiators $\Delta\varphi$ shall not exceed 20°, and that the ratio of the ordinates of the amplitude frequency characteristics lie within the limits of 0.8 to 1.2. The derivative coefficient T_d of the real differentiator is bounded. Therefore, in addition to the working range of the frequency characteristic, the magnitude of T_d for the given differentiator is determined.

All differentiating devices can be divided into two types: differentiators in which the operation of differentiation is modeled (analog differentiators) and differentiators in which the operation of differentiation is replaced by the determination of the increment of the input quantity during finite intervals of time (we agree to call such devices discrete differentiators). For the differentiation of slowly varying signals, discrete differentiators turn out to be competitive with analog differentiators.

Analog Differentiators

In this section we shall consider two basic types of electronic differentiators, designed for the differentiation of dc signals, and an electric differentiator in the form of a servosystem with proportional velocity control. The basic elements of electronic dc differentiators are RC circuits* and amplifiers. After determining the working range of the frequency characteristics in the linear treatment of these differentiators, one must take into account the effect of nonlinearities in the static characteristics of amplifiers and motors on the corresponding frequency characteristics.

Differentiation of ac signal envelopes by means of passive circuits is, due to large errors, efficacious only for rapidly varying signals (cf., for example, [1]).

A. Differentiator in the Form of a Differentiating RC Circuit with an Amplifier Connected at Its Output†

The transfer function of an inherently differentiating circuit with account taken of the impedance of the leakage condenser‡ is written in the form

$$\frac{\overline{U}_{out}(p)}{\overline{U}_{in}(p)} = \frac{Tp + R/R_{le}}{Tp + 1 + R/R_{le}}, \tag{1}$$

where T = RC is the time constant (coinciding with the derivative coefficient when the amplifier is not connected in), R/R_{le} is the ratio of the circuit's active impedance to the impedance of the leakage condenser, $\overline{U}_{in}(p)$ and $\overline{U}_{out}(p)$ are the transforms of the input and output voltages, respectively.

In Fig. 1 there are constructed the relative amplitude and the phase frequency characteristics. The upper boundary of the working range of the frequency characteristics is due to the presence in (1) of the aperiodic link (in accordance with the conditions adopted earlier for the phase frequency characteristic, $(\Omega T)_{max} = 1$). The lower boundary of the working range is due to the static term in the ampli-

*In differentiating slowly varying signals, an RC circuit is more convenient than an RL circuit since, by means of an RC circuit, one can easily obtain a larger time constant for a lower sensitivity to noise.

† An RC differentiator is described, for example, in [2].

‡ To increase the time constant one uses a condenser with a large capacity, in which the leakage impedance can be commensurable with R.

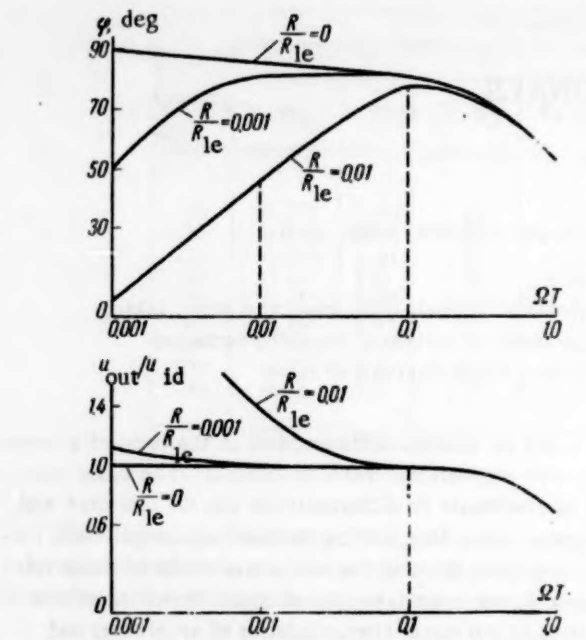

Fig. 1. Phase frequency and relative amplitude frequency characteristics of a differentiating RC link.

fier's transfer function. With this, the magnitude of $(\Omega T)_{min}$, as is clear from the phase frequency characteristics constructed for various R/R_{le}, decreases proportionately to the decrease in R/R_{le}. However, for each type of condenser there exists an absolute magnitude of Ω_{min} which depends only on the coefficient of impedance of the leakage condenser $k_{le} = R_{le}C$.

By substituting $C = k_{le}/R_{le}$ in (1) we obtain

$$\frac{\bar{U}_{out}(p)}{\bar{U}_{in}(p)} = \frac{R/R_{le}(k_{le}p+1)}{R/R_{le}k_{le}p+1+R/R_{le}}. \tag{2}$$

It is clear from (2) that, for low frequencies, the effect of the static term on the transfer function depends only on the magnitude of k_{le}. For example, for $\Delta\varphi = 20°$ and $k_{le} = 3000$ μf·meg, $\Omega_{min} = 0.001$ cps.

It is clear from (1) that the differentiating circuit's time constant coincides with the derivative coefficient. To increase this latter, we connect an amplifier to the differentiating circuit's output.

If we assume that the amplifier has a linear static characteristic, then the derivative coefficient is increased by a factor of k, where k is the amplifier's gain**.

The size of k is limited by the increase, at the amplifier's output, of parasitic signals due to internal noise, and also by the decrease in range of variation of the input signal corresponding to the linear portion of the amplifier's static characteristic.

The effect of internal noise on the working range of the frequency characteristic can be estimated by the ratio y_p of the parasitic signal u_p††, as a fraction of the input signal, to the maximum value of the useful signal during one period (for $u_{in}(t) = B \sin \Omega t$):

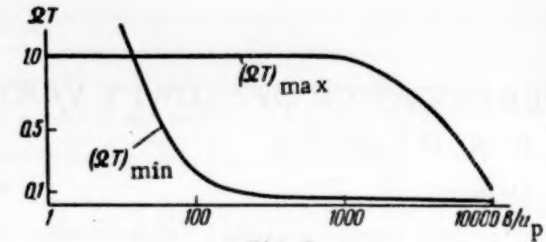

Fig. 2.

$$y_p = \frac{u_p}{B}\frac{1}{\Omega T}. \tag{3}$$

where B is the amplitude of the input signal.

On the other hand, for operation on the linear portion of the amplifier's static characteristic, it is necessary that the following inequality hold for $u_{in.1}$

$$B\Omega T \leqslant u_{in.1}. \tag{4}$$

The aforementioned peculiarities of amplifier operation will not affect differentiator operation if, in the given frequency range, as B varies within given limits, y_p remains much less than unity and satisfies (4). Figure 2 shows the graphs constructed for $(\Omega T)_{min} = f_1(B/u_p)$ and $(\Omega T)_{max} = f_1(B/u_p)$ for $y_p = 0.05$ and $u_{in.1}/u_{1p} = 1000$ (for $k = 1000$). It is clear from these graphs that, because of the limitation on the magnitude of $(\Omega T)_{min}$ and the limits of variation of the input signal's amplitude, one can increase the derivative coefficient only by a factor of 1000.

B. Differentiator in the Form of an Amplifier with Aperiodic Feedback

This differentiating device consists of an amplifier, shunted by an aperiodic feedback path, and a subtraction block in which the input signal is subtracted from the amplifier's output signal (Fig. 3). The subtraction block is included so as to eliminate dc components from the amplifier's output signal. If the amplifier's characteristics

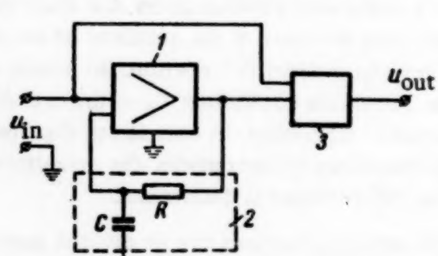

Fig. 3. Block schematic of a differentiator in the form of an amplifier with aperiodic feedback. 1 is the amplifier, 2 is the aperiodic feedback link, and 3 is the subtraction block.

**It is assumed that the amplifier's input impedance is much greater than the RC circuit's output impedance.

†† We understand by u_p an averaged value of the parasitic signal, the magnitude of which depends on the amplifier circuit.

are assumed to be linear, then the transfer function of such a system will have the form

$$\frac{\overline{U}\text{out}\,(p)}{\overline{U}\text{in}\,(p)} =$$

$$= \frac{(1+k_1)\,k}{(k+k_1+k)}\left(\frac{T}{1+k_1}\,p+1\right)\frac{1}{\dfrac{Tp}{k+k_1+1}+1} - 1, \tag{5}$$

where $k_1 = R/R_{in.a}$ and $R_{in.a}$ is the amplifier's input impedance.

From (5) one may show that, to eliminate the dc component from u_{out}, $\underline{k}$ and k_1 must satisfy the relationship

$$k_1 = \frac{1}{k-1}. \tag{6}$$

By substituting k_1 from (6) in (5), we get

$$\frac{\overline{U}_{out}\,(p)}{\overline{U}_{in}\,(p)} = \frac{(k-1)^2}{k^2}\,Tp\left(\frac{Tp}{k+k/(k-1)}+1\right)^{-1}. \tag{7}$$

Figure 4 gives the amplitude and phase frequency characteristics of the system. It is clear from these graphs that, as $\underline{k}$ increases, the magnitude of $(\Omega T)_{max}$ increases. However, in the given case, the derivative coefficient equals the time constant of the RC circuit. From the expressions $k_1 = R/R_{in.a}$, $k_1 = 1/(k-1)$, and $T = RC$ for $k \gg 1$, we can obtain

$$T \approx \frac{R_{in.\,a}}{k}C. \tag{8}$$

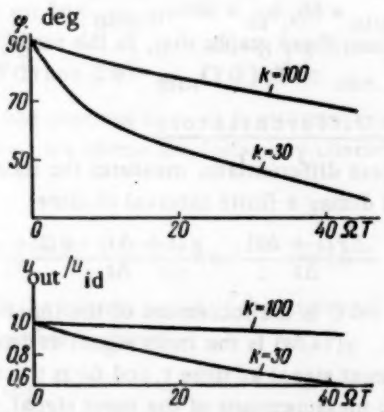

Fig. 4. Phase frequency and relative amplitude frequency characteristics of the differentiator in the form of an amplifier with aperiodic feedback.

Thus, as k increases, T decreases. This decrease can be compensated by increasing the capacity of the condenser until R_{1e} becomes commensurable with $R_{in.a}$. In practice, even for an amplifier with $R_{in.a}$ = 10 meg, for k = 100 and C = 500 μf, T = 50 sec. Thus, in a linear system there is a limitation on the size of T_p. From whence, the magnitude of $(\Omega T)_{max}$ in practice equals 40.

In a linear system no limitation on $(\Omega T)_{min}$ was obtained, since the dc component due to the condenser leakage was compensated here in the subtraction block. However, in practice the subtraction block has a finite accuracy in implementing the operation of subtraction. We assume that the error with this comprises 0.1% of the dc component of u_{in} ‡‡. As in the previous case, we define the ratio

$$y_s = \frac{0.001}{\Omega T}. \tag{9}$$

If, as before, we set $y_s = 0.05$, we get that $(\Omega T)_{min} = 0.02$.

The upper boundary of the working range does not depend on the amplitude of the input signal if $u_{in,1}/B \geq 1$, where this holds for any frequency***.

C. Servosystem with Proportional Velocity Control as a Differentiating Device

The simplified schematic of the servosystem with proportional velocity control is shown in Fig. 5a. The system consists of electronic amplifier 1, low-power motor 2 controlled by the electronic amplifier, reducer 3, and feedback position transducer 4. In a first approximation, we assume that all links of the system have linear characteristics, and that the time constants of the amplifier and motor may be neglected †††. We write the equations of such a system in the form:

$$u_m = k_a\varepsilon. \tag{10}$$

$$v = k_m u_m \tag{11}$$

$$v_r = \frac{v}{k_r}. \tag{12}$$

$$\overline{U}_{fb}(p) = \frac{k_{fb}\overline{V}_r(p)}{p}. \tag{13}$$

$$\varepsilon = u_{in} - u_{fb} \tag{14}$$

where ε is the error signal, u_d is the voltage on the control winding, $\underline{v}$ is the motor speed prior to the reducer, v_r is the motor speed after the reducer, $\overline{V}_r(p)$ is the transform of v_r, $\overline{U}_{fb}(p)$ is the transform of the feedback position voltage, u_{in} is the input signal voltage, k_a is the amplifier gain, k_r is the reduction factor, k_{fb} is the position feedback gain, showing the change in u_{fb} for one rotation.

From (10)-(14) we obtain

$$\frac{\overline{V}(p)}{k_a k_m} = \frac{Tp}{Tp+1}\,\overline{U}_{in}\,(p), \tag{15}$$

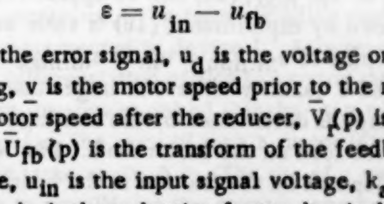

Fig. 5a.

‡‡ It is obvious that, for k = 100, the amplifier's background noise will be less than the error from subtraction.

*** This can be verified by defining the signal at the amplifier's input as the difference between $u_{in} = B\sin\Omega t$ and the feedback signal.

††† The permissibility of this last assumption will be explained below.

where $\bar{U}_{in}(p)$ is the transform of $u_{in}(t)$, $\bar{V}(p)$ is the transform of $v(t)$ and $T = k_r/k_a k_m k_{fb}$.

Expression (15) coincides in form with (1) for the case when $R/R_{1e} = 0$. For the measurement of slowly varying signals, as was established earlier, the value of T should be chosen large (for $R/R_{1e} = 0$). Therefore, for low-power motors, the coefficient $k_r/k_a k_m k_{fb}$ will be larger than the electrical and electromechanical time constants; the working range of frequency characteristics can be found from Fig. 1 (for $R/R_{1e} = 0$, $(\Omega T)_{min} = 0$, $(\Omega T)_{max} \approx 1$).

As the servosystem's output signal, one may use ϵ, u_m, and $\underline{v}$ (a voltage proportional to $\underline{v}$). Measurement of ϵ is equivalent to the case of a passive RC network with an ideal capacitor, while measurement of u_m and $\underline{v}$ is equivalent to the case when an amplifier is connected at the output of the RC network.

The dynamics of the actual servosystem differ from those of the linear system principally because of the nonlinearities of the static motor characteristics and free play in the reducer. It was shown in [3] that the function $v = F(u_m)$ for low-power asynchronous motors has the form

$$\frac{v}{v_{max}} = \frac{2\dfrac{u_m - u_{st}}{u_{m.\,max}}}{1 + \dfrac{u_m - u_{st}}{u_{m.\,max}}}. \qquad (16)$$

where v_{max} is the maximum motor speed, $u_{m.max}$ is the maximum voltage on the control winding, u_{st} is the voltage corresponding to starting.

For $u_m \ll u_{m.max}$, (16) can be approximated by (11). As shown by experiments, (16) is valid as u_m varies within the limits of $u_{m.min} \le u_m \le u_{m.max}$. By $u_{m.min}$ we understand a quantity somewhat larger than u_{st}. The starting voltage is determined by the starting torque, which depends on the load on the motor shaft. The load torque, in its turn, depends on the friction in the bearings, which is not a strictly constant quantity. Therefore, the motor shaft can turn continuously, starting with some velocity v_{min} which we agree to call the minimum velocity of continuous rotation.

If the voltage on the motor winding lies within the limits of $u_{st} \le u_m \le u_{m.min}$ there then occurs a discontinuous mode of motor operation. For $u_{in} = B \sin \Omega T$, a definite portion of the period will be comprised of discontinuous system operation. We may reflect this idiosyncracy of actual servosystem operation by the ratio y_m (similar to the ratio y_p):

$$y_m = \frac{u_{in.\,min}}{B}\,\frac{1}{\Omega T}. \qquad (17)$$

where $u_{in.min}$ is the voltage at the amplifier input which corresponds to $v_{m.min}$; (17) is analogous in form to (3) but, in practice, $u_{in.min}$ is significantly larger than u_p.

For input signals with shallow slopes, the free play in the reducer manifests itself in the servosystem. Because of the presence of the free play, as the sign of the slope of $u_{in}(t)$ changes, the speed of the motor will not correspond to the slope of the input signal during the time that the free play is being taken up. It can be shown (cf. Appendix 1) that the ratio of the motor speed at the moment when the free play has been taken up to the maximum speed during the period (for $u_{in}(t) = B \sin\Omega t$) equals

$$y_f = 1.65\left(\frac{m_f\, k_{fb}}{B}\right)^{\frac{2}{3}} \frac{1}{\sqrt[3]{\Omega T}}. \qquad (18)$$

where m_f is the free play in the first stage of the reducer.

The upper limit to the working range can be given in the form of (4), where by $u_{in.1}$ we shall understand u_{in} corresponding to the boundary of approximation of nonlinear relationship (16) by the linear static characteristic of (11).

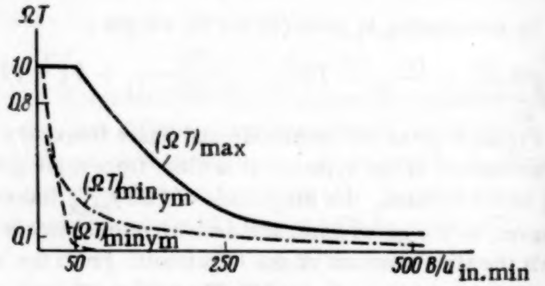

Fig. 5b.

Figure 5b shows the curves constructed for $(\Omega T)_{min}$‡‡‡ and $(\Omega T)_{max}$ as functions of $B/u_{in.min}$ for $y_m = y_f = 0.05$, $u_{in.1}/u_{in.min} = 50$, $k_{fb} = 500 u_{in.min}$ and $m_f = 0.0001$. It is clear from these graphs that, in the range $20 \le B/u_{in.min} \le 50$, $(\Omega T)_{min} \approx 0.5$ and $(\Omega T)_{max} \approx 1$.

Discrete Differentiators

A discrete differentiator measures the increment of the input signal during a finite interval of time:

$$\frac{\Delta g(t+\Delta t)}{\Delta t} = \frac{g(t+\Delta t) - g(t)}{\Delta t}. \qquad (19)$$

where $\Delta g(t+\Delta t)$ is the increment of the input signal at time $t+\Delta t$, $g(t+\Delta t)$ is the input signal at time $t+\Delta t$, $g(t)$ is the input signal at time $\underline{t}$, and Δt is the time interval between measurements of the input signal.

A discrete differentiator may be constructed either on the principle of quantized input signal g(t) or on the principle of quantized time. In measuring the speed of change of a slowly varying input signal, Δg will be small, so that it is more convenient in practice to use the second principle (the value of Δt can be chosen arbitrarily).

The dynamic characteristics of discrete differentiators[1] depend on the method of measuring the increments of the input signal. One can cite two basic methods of measuring Δg:

‡‡‡ In correspondence with (17) and (18), two curves are given for $(\Omega T)_{min}$. For a given value of $B/u_{in.min}$, the lower boundary of the frequency characteristic corresponds to the larger ordinate.

[1] It is understood that we are speaking of discrete differentiators with quantized time.

1. Comparison of the instantaneous values of $g(t+\tau)$ and $g(t)$, where τ is a fixed interval between two measurements [cf. (19)].

In (19) we replace the continuous argument t by the discrete argument $n\tau$ (n corresponds to the moments at which Δg is measured). Then, letting $g(t) = B \sin \Omega t$, we obtain

$$\frac{\Delta g_1 (n\tau + \tau)}{B\tau} = \frac{1}{\tau} [\sin \Omega (n+1)\tau - \sin \Omega n\tau]. \quad (20)$$

By transforming (20), we find that

$$\frac{\Delta g_1 (n\tau + \tau)}{B\tau} = \frac{2}{\tau} \sin \frac{\Omega\tau}{2} \cos\left(\Omega n\tau + \frac{\Omega\tau}{2}\right). \quad (21)$$

Expression (21) can be considered as the system's frequency characteristic for a discrete argument. This means physically that (21) shows the difference $\Delta g_1/B\tau$ from the ideal derivative only for the moments of time $n\tau$. The relative amplitude and phase frequency characteristics are given in Fig. 6 (curve 1). As the argument of the frequency characteristics, we have here chosen $\Omega\tau$. It is clear from these graphs that the working range of relative values of $\Omega\tau$ has just the upper boundary of $(\Omega\tau)_{max} \approx 0.6$.

2. Comparison of the results of integrating the input signal over the intervals of time $[t, t+\tau]$ and $[t+\tau,\ t+2\tau]$, respectively:

$$\frac{\Delta g_2}{B\tau} = \frac{1}{\tau}\left[\int_{t+\tau}^{t+2\tau} g(t)\,dt - \int_{t}^{t+\tau} g(t)\,dt\right]. \quad (22)$$

As in the previous case, it is assumed that $g(t)$ varies sinusoidally. We obtain the frequency characteristic in the form

$$\frac{\Delta g_2 (\varphi + 2\Omega\tau)}{B\tau} = \frac{4}{\Omega\tau} \sin^2 \frac{\Omega\tau}{2} \cos(\varphi + \Omega\tau), \quad (23)$$

where φ is the current value of the argument (analogous to $n\tau$ in number 1 above).

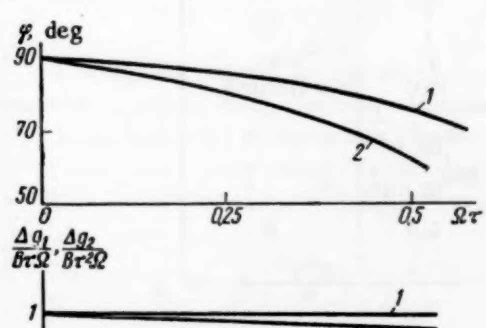

Fig. 6. Phase frequency and relative amplitude frequency characteristics of discrete differentiators. 1 is for a discrete differentiator of the first type, 2 is for a discrete differentiator of the second type.

The relative amplitude and phase frequency characteristics are given in Fig. 6 (curve 2).[2] It is clear from them that the upper boundary of the working range has been lowered, and that $(\Omega\tau)_{max} \approx 0.45$.

It is clear from (21) and (23) that the derivative coefficient (for small $\Omega\tau$) equals, respectively, 1 and τ.[3] In the first case, T_d can be increased only by connecting an amplifier at the output of the discrete differentiator and, in the second case, at the cost of increasing τ. However, connecting an amplifier reduces the working range (cf. above), and increasing τ decreases Ω_{max} (Fig. 6). Moreover, implementing the necessary operations to determine increments on the basis of analog technology leads to large errors. For example, in the discrete differentiators described in [4-6], there will be large dc components at the output due to the nonideal nature of the elements. The transition to digital technology, as will be shown below, retains the working range of the frequency characteristics with an arbitrarily large derivative coefficient.

We consider a discrete digital differentiator. To measure the results of integrations over the interval τ, it is convenient to present the input signal $g(t)$ in the form of frequencies of trains of pulses (pulse-frequency modulation):

$$f(t) = f_0 + l g(t), \quad (24)$$

where f_0 is the minimum frequency, corresponding to $f(t) = 0$, and l is the slope of the analog-digital transformation[4].

Figure 7 shows the block schematic of a discrete digital differentiator. The transformation corresponding to (24) is implemented in device 1. As a reversible counter for the differentiation of slowly varying signals, one may use the low-power pulsed step-by-step motor 2. In switching phase, the rotor of such a motor is rotated by one and the same angle $\Delta\varphi$. The use of a pulsed motor, in addition to simplifying the counter scheme, permits the implementation of a digital-analog transformation by means of inductive transducer 3. The intervals of addition and subtraction are given by synchronous motor 4. The control scheme in 5 provides: a) switching of motor phase

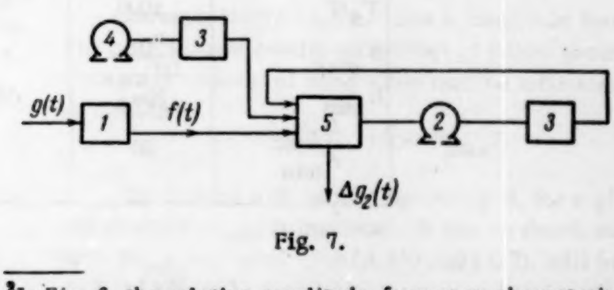

Fig. 7.

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[2] In Fig. 6, the relative amplitude frequency characteristic was constructed in the dimensionless units $\Delta g_2/B\tau^2\Omega$ instead of $\Delta g_2/B\tau\Omega$.

[3] One can convince oneself of this by expanding $\sin(\Omega\tau/2)$ in a series and retaining just the first term of the expansion.

[4] Such a transformation can be implemented either by means of a special device or by the use of transducers with digital outputs.

with frequency $f(t)$; b) reversal of the direction of phase switching upon transition from addition to subtraction; c) transmission to the differentiator's output of the measured increment of the input signal; d) reset of the system to its initial state.

We now consider what the conditions are which must be satisfied by the digital differentiator's parameters in order to provide the necessary working range of the frequency characteristic (Ω_{min} to Ω_{max}).

The integration time τ can be found, for a given upper boundary of the working range of the frequency characteristic, from the phase frequency characteristic of an ideal discrete differentiator in which $(\Omega\tau)_{max}$ equals 0.45 (Fig. 6, curve 2):

$$\tau = \frac{0.45}{\Omega_{max}}. \tag{25}$$

In a digital system, Δg will be expressed in the form of some number $\underline{m}$ of pulses. The general expression for $\underline{m}$, as well as the expressions for m_{min} and m_{max} for a given frequency Ω and amplitude B in the case of a sinusoidal input signal, are derived in Appendix 2. The quantity m_{min} characterizes the minimum accuracy in measuring Δg over one period of signal variation. By considering m_{min} to be a given quantity, we obtain from (A.10), after substituting the value of τ from (25), $\Omega = \Omega_{min}$ and $B = B_{min}$, the expression for

$$l = \frac{8m_{min}}{B_{min}}\Omega_{max}\left(\frac{\Omega_{max}}{\Omega_{min}}\right)^2. \tag{26}$$

With this, the maximum number of pulses necessary to make the transformation from digital form to analog will equal, on the basis of (A.11), $m_{max} = (2B_{max}/B_{min}) \times m_{min}(\Omega_{max}/\Omega_{min})^2$.

Finally, by taking into account the optimal relationship between f_0 and lB(cf., Appendix 2), we find from (A.12) the expression for the maximum capacity of the counter:

$$n_{max} = \frac{8B_{max}}{B_{min}} m_{min}\left(\frac{\Omega_{max}}{\Omega_{min}}\right)^2.$$

Of all the parameters of the digital differentiator, it is the quantity m_{max} which may be limited in practice since, with increasing m_{max} for a given type of digital differentiator, there is a decrease in the magnitude of $\Delta\varphi$ which, in practice, has a lower limit. Therefore, for given m_{min} and m_{max}, a digital differentiator has a bounded working range. For example, $\Omega_{max}/\Omega_{min} = 22$ for $m_{min} = 3$, $m_{max} = 10{,}000$ and $B_{max}/B_{min} = 5$.

In contradistinction to the second type of discrete differentiator which was considered earlier, in the given digital differentiator T_d depends, not only on τ, but also on the proportionality factor between the angle of rotation of the transducer's shaft and the change in voltage at the output of the induction transducer. Here, the proportionality factor does not depend on the working range, and T_d may in fact be made as large as desired.

Comparison of Dynamic Characteristics of Differentiating Devices

The results of a comparison of the three types of analog differentiating devices and the digital discrete differentiator are presented in the following table

	RC network plus ampl.	Ampl. with aperiodic feed-back;	Servo system	Digital differentiator
Ω_{min} 1/sec	0,001	0.001	$\frac{0.4}{T}$	Not limited
Ω_{max} 1/sec	$\frac{1}{T}$	0.8	$\frac{1}{T}$	20 Ω_{min}
T_d/T	1000	100	1000	Not limited
B_{max}	$\frac{u_{in.1}}{\Omega T_{max}}$	$u_{in.1}$	$u_{in.1}$	» »
B_{min}	$20u_p$	$20\ u_p$	$20\,u_{in.min}$	» »
$\frac{B_{max}}{B_{min}}$	50	50	2.5	5

It is clear from the table that only the servosystem and discrete digital type differentiators have no limit on the smallness of the quantity Ω_{min}. In the digital differentiator, moreover, the derivative coefficient can be made arbitrarily large. The sole superior feature of the analog differentiator is its greater simplicity vis-a-vis the digital system. However, for the differentiation of slowly varying signals, one can use a pulsed motor in the digital differentiator, which significantly simplifies the entire system.

Summary

Analog differentiating devices with large working ranges (an amplifier at the output of an RC network, or

an amplifier with aperiodic feedback) have a limited magnitude of Ω_{min} and of the derivative coefficient. In an electromagnetic differentiator, the quantity Ω_{min} can be made arbitrarily small, and $\Omega_{max}/\Omega_{min}$ can be made less than in electronic differentiators, but the derivative coefficient is of the same order of magnitude.

It is advantageous to construct discrete differentiators in the form of digital systems. A digital differentiator permits one to have an arbitrarily small value of Ω_{min} and an arbitrarily large derivative coefficient.

Appendix 1

For simplicity of computation, we assume that the motor characteristic $v = F(u_m)$ is linear. The free play in the reducer manifests itself when the sign of the input signal's slope changes. We now determine the motor speed at the moment when the free play has all been taken up, given a sinusoidal input signal:

$$v_f = k_a k_m B(1-\cos\Omega t_f) \approx k_a k_m B\frac{\Omega^2 t_f^2}{2}, \qquad (A.1)$$

where v_f is the motor speed at the moment when the free play has all been taken up. Knowing the shaft rotation angle φ_f corresponding to taking up of the free play for a given reducer transfer number, we can, by integrating (A.1) and solving the resulting equation for t_f, determine this quantity in the form

$$t_f = \sqrt[3]{\frac{6\varphi_f}{k_y k_f B\Omega^2}}. \qquad (A.2)$$

By substituting (2) in (1), we find the expression for v_f in terms of φ_f:

$$v_f = k_a k_m B\cdot 1.65\left(\frac{\varphi_f\Omega}{k_y k_m B}\right)^{\frac{2}{3}} \qquad (A.3)$$

We now find the ratio of the maximum speed during one period at a given frequency to the velocity v_f:

$$\gamma_f = \frac{1.65}{\Omega T}\left(\frac{\varphi_f\Omega}{k_a k_m B}\right)^{\frac{2}{3}} \qquad (A.4)$$

The angle of rotation for free play take-up can be presented as a product $m_f k_r$, where m_f corresponds to the free play in the first stage of the reducer (the free play in the other stages has a lower value). By multiplying the numerator and denominator in (4) by k_{fb}, we obtain finally

$$\gamma_f = 1.65\left(\frac{m_f k_{fb}}{B}\right)^{\frac{2}{3}}\frac{1}{\sqrt[3]{\Omega T}}. \qquad (A.5)$$

Appendix 2

The length of the period between two neighboring pulses for a sinusoidal input signal with arbitrary initial phase $\varphi\,[g(t) = B\sin(\Omega t+\varphi)]$ is defined by the equation

$$\int_0^{T_i}[f_0 + lB\sin(\Omega t+\varphi)]\,dt = 1, \qquad (A.6)$$

where T_i is the length of the ith period.

By integrating (A.6) we get

$$f_0T_i - \frac{lB}{\Omega}\cos(\Omega T_i+\varphi) + \frac{l}{\Omega}\cos\varphi = 1.$$

We now find the expression defining the number n of accumulated pulses during the interval $[\varphi, \varphi+\tau]$:

$$f_0T_1 - \frac{lB}{\Omega}\cos(\Omega T_1+\varphi) + \frac{lB}{\Omega}\cos\varphi = 1,$$

$$f_0T_2 - \frac{lB}{\Omega}\cos(\Omega T_1+\Omega T_2+\varphi) + \frac{lB}{\Omega}\cos(\varphi+\Omega T_1) = 1, \qquad (A.7)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_0T_n - \frac{lB}{\Omega}\cos(\Omega T_1+\ldots+\Omega T_n+\varphi)$$

$$+\frac{lB}{\Omega}\cos(\varphi+\Omega T_1+\ldots+\Omega T_{n-1}) = 1,$$

$$f_0\sum_i^n T_i - \frac{lB}{\Omega}\cos\left(\Omega\sum_i^n T_i+\varphi\right) + \frac{lB}{\Omega}\cos\varphi = n,$$

$$f_0\tau - \frac{lB}{\Omega}\cos(\Omega\tau+\varphi) + \frac{lB}{\Omega}\cos\varphi = n.$$

One may show analogously that, during the interval $[\varphi+\tau, \varphi+2\tau]$, there will, in the general case, be accumulated $(n+m)$ pulses:

$$f_0\tau - \frac{lB}{\Omega}\cos(2\Omega\tau+\varphi) + \frac{lB}{\Omega}\cos(\varphi+\Omega\tau) = n+m. \qquad (A.8)$$

By subtracting (A.7) from (A.8), we get

$$m = \frac{lB}{\Omega}\,[2\cos(\Omega\tau+\varphi) - \cos(2\Omega\tau+\varphi) - \cos\varphi]. \qquad (A.9)$$

For $\varphi = 3\pi/2$, the number m is a minimum:

$$m_{min} = \frac{4lB}{\Omega}\sin^2\frac{\Omega\tau}{2}\sin\Omega\tau \approx lB\tau(\Omega\tau)^2. \qquad (A.10)$$

For $\varphi = \pi$, m is maximum. The approximate maximum value of m is

$$m_{max} \approx lB\tau(\Omega\tau). \qquad (A.11)$$

Correspondingly, (A.7) attains a maximum for $\varphi = \pi/2$. The approximate maximum number of pulses accumulated during one interval of integration may be estimated by the expression

$$n_{max} \approx \tau(f_0 + B_{max}l). \qquad (A.12)$$

The counter will be used optimally if, for a given n, the quantity m_{min} is maximal. It can be shown that the ratio m_{min}/n, found from (A.10) and (A.7), will be maximal as $lB/f_0 \to 1$.

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CHOICE OF FREQUENCY RANGE FOR INDUSTRIAL PULSE-FREQUENCY TELEMETRY SYSTEM DEVICES

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The paper considers the effect of frequency range of pulse-frequency telemetry system devices on the errors and noise resistance of transmission, speed of action, and receiver dynamics.

The following factors influence the choice of frequency range of pulse-frequency telemetry system devices:

1) telemetry error as a function of frequency range;

2) noise resistance of signal transmission along the communications channel, and the effect of noise from the 50 cps power supply;

3) indicating instrument dynamics and speed of system action;

4) bandwidth of the secondary channel.

1. Telemetry Error as a Function of Frequency Range

As a function of the ratio of the frequencies which correspond to zero value and to the nominal value of the measured parameter, the error α in transforming this parameter to a proportional frequency in the transmitter gives rise to the telemetry error [1]

$$\alpha_{te} = \frac{\alpha}{1 - f_0/f_N}. \qquad (1)$$

where f_0 and f_N are the frequencies corresponding to the zero value and to the nominal value, respectively, of the parameter being measured.

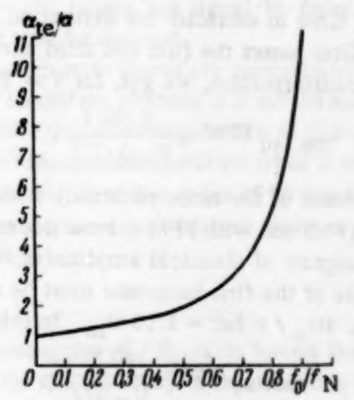
Analogously, the error β arising in the receiver when the frequency is transformed to a proportional current gives rise to the error

$$\beta_{te} = \frac{\beta}{1 - f_0/f_N}. \qquad (2)$$

The ratio of the errors is

$$\frac{\alpha_{te}}{\alpha} = \frac{\beta_{te}}{\beta} = \frac{1}{1 - f_0/f_N}. \qquad (3)$$

The figure shows the relationship of the ratio of errors, α_{te}/α or β_{te}/β, to f_0/f_N. As is obvious from (3) and from the graphs on the figure, the telemetering error decreases as the ratio f_0/f_N decreases.



Dependence of the ratio of the telemetering errors to the ratio of the frequencies corresponding to the zero value and to the nominal value of the parameter being measured.

2. Noise Resistance of Signal Transmission along the Communications Channel and the Effect of Noise from the 50 cps Power Supply

With weak fluctuation noise, the error is determined by the distortion of the pulse edges of the transmitted signals. This error, as a rule, is a very small quantity.

An analysis of the noise resistance of telemetry signal transmission along a channel with weak fluctuation noise, based on the theory of potential noise resistance, was made by V. A. Kashirin and G. A. Shastova [2]. However, in considering pulse-frequency modulation with a constant mark-space ratio PFM II, the authors of [2] analyzed the transmission of triangular pulses with amplitude $2U_m$. The transmission of triangular pulses negates all the advantages of PFM II, and does not correspond to actual conditions.

The noise resistance of PFM II to fluctuation noise can be determined by expanding rectangular pulses in a Fourier series. The pulse frequency in PFM II can be presented in the following manner as a function of the measured parameter λ:

$$f(\lambda) = f_1 + \frac{\Delta f}{2}\lambda, \qquad (4)$$

where f_1 is the mean pulse frequency, and $\Delta f/2 = (f_N - f_0)/2$ is the frequency deviation.

The expression for a rectangular signal of amplitude $2U_m$, after expansion in a Fourier series, can be written in the form

$$A(\lambda, t) = U_m + \frac{4U_m}{\pi}\left[\sin 2\pi\left(f_i + \frac{\Delta f}{2}\lambda\right)t + \frac{1}{3}\sin 6\pi\left(f_i + \frac{\Delta f}{2}\lambda\right)t + \ldots\right]. \quad (5)$$

The mean square error in transmitting the parameter λ_0 due to fluctuation noise is defined by the expression [2]

$$\delta_{me.\,sq}(\lambda) = \frac{\sigma}{2\sqrt{2}\sqrt{\int\limits_0^T \left[\frac{\partial A(\lambda, t)}{\partial\lambda}\right]^2_{\lambda=\lambda_0} dt}}, \quad (6)$$

where σ is the specific intensity of the noise in $v(cps)^{1/2}$ and T is the time to establish the indication.

If the filter passes the first and third harmonics then, after some transformation, we get, for $T \gg 2\pi/f_i$,

$$\delta_{me.\,sq}\,PFM = \frac{0.186\,\sigma}{U_m \Delta f T^{3/2}}. \quad (7)$$

Comparisons of the noise resistance with frequency modulation (FM) and with PFM II must necessarily be carried out for signals of identical amplitudes. With this, the amplitude of the first harmonic must be considered equal, not to $4U_m/\pi$ but $\sim 1.15\,U_m$. In this case,

$$\delta_{me.\,sq}\,PFM = \frac{0.22\,\sigma}{U_m \Delta f T^{3/2}}. \quad (8)$$

For the same amplitude, the error with FM is (cf., [2])

$$\delta_{me.\,sq}\,FM = \frac{0.28\,\sigma}{U_m \Delta f T^{3/2}}. \quad (9)$$

Consequently, if the use of filters provides transmission of the first and third harmonics, with PFM II, then with this the mean square error from fluctuation noise will be less than with FM with filters of the same passband. Low-frequency pulse-frequency systems for transmitting first and third harmonics require the same (and sometimes a smaller) frequency range than the existing frequency telemetry systems.

N. V. Pozin [3] carried out a determination of the error with PFM due to the relatively strong noise when the basic signal distortions are due to the breakdown into pulses and pauses, formulas being derived for approximate determinations of the mean and the mean square errors as functions of the probability of a unit distortion, the filter passband $2F_{pb}$, the frequencies corresponding to the zero and nominal values of the measured parameter, and the establishment time of the indicating instrument:

$$\delta_{me} = \frac{2P}{\Delta f^2}\left[\left(\frac{F_{pb}}{2} - f_0\right)^2 - \left(f_N - \frac{F_{pb}}{2}\right)^2\right], \quad (10)$$

$$\delta_{me.\,sq} = \sqrt{\frac{2P}{T\Delta f^3}\left[\left(\frac{F_{pb}}{2} - f_0\right)^2 - \left(f_N - \frac{F_{pb}}{2}\right)^2\right]}, \quad (11)$$

where P is the probability of unit distortion.

The minus sign between the parenthetical expressions in (10) and (11) is due to the fact that, with PFM II, the subcarrier frequency filter must pass at least the third harmonic and, consequently, $F_{pb} \geq 3 f_N$. This latter determines the choice of the frequency range for PFM II. Otherwise, the probability of unit distortion would not be the same for all pulse durations, as was assumed [3] in the derivation of (10) and (11).

The table gives the results of computing δ_{me} and $\delta_{me.sq}$ for the telemetry devices of the pulse-frequency devices of TsLĖM Mosenergo, IAT AN SSSR and TsNIIKA for operation with filters of passbands of 40 cps (developed by IAT AN SSSR), 80 and 140 cps (telegraph filters), starting from the assumption that the formulas for determining δ_{me} and $\delta_{me.sq}$, to a first approximation, reproduce the state of affairs when the transition is made from PFM II to FM [3].

In the table, P_{40}, P_{80}, and P_{140} denote, respectively, the probability of unit signal distortion for filter passbands of 40, 80, and 140 cps, with identical signal voltages in all cases.

It is clear from the tabulated data that, for the standard telegraph filters, the very best frequency range of those considered is the range from 4 to 20 cps.

In addition to the considerations already mentioned, the noise from the 50 cps power supply (line) has an effect on the choice of frequency range for the devices of frequency and pulse-frequency telemetry systems. The presence of the frequencies of 25, 50, and 100 cps in the frequency range leads to a strengthening of the effect of noise from the power supply on the operation of the receiving device in the neighborhood of the points cited. In the transmission of the telemetered quantity at a frequency of 25 cps there appears, at the receiving side after some nonlinear transformations, a component of the first harmonic of the industrial frequency (50 cps) which is essentially influenced by the power supply.

Since it is desirable to have the ratio f_0/f_N as low as possible, so as to provide accuracy of transmission, one should choose, for the devices of pulse-frequency telemetry systems, a nominal frequency less than 25 cps. If one takes into account the possible oscillations of the line frequency (under trouble conditions, the frequency in energy systems falls to 46 cps) and the fact that, in using standard telegraph filters with 140 cps pass bands, it is desirable to transmit the third harmonic, then one should take, as the very best nominal frequency, one approximately equal to 20 cps.

3. Dynamics of the Indicating Instrument and the System's Speed of Action

The least (zero) frequency of teletransmission is determined by the highest frequency of variation of the measured parameter, F_m, and, for analog reception, by the indicating instrument dynamics. Since industrial telemetry devices are used, basically, for the measurement of relatively slowly flowing processes, the condition

$$f_0 \geqslant 2F_m, \quad (12)$$

Error	Filter passband, cps	Frequency range		
		$f_0=1$ cps f_N 10cps (TsLEM Mosénergo) $T=4$ sec	$f_0=5$ cps $f_N=15$ cps (IAT AN SSSR) $T=4$ sec	$f_0=4$ cps, $f_N=20$ cps (TsNIIKA) $T=4$ sec
δ_{me}	40	$2\ P^{40}$	P_{40}	$1.06\ P_{40}$
$\delta_{me.\ sq.}$	40	$0.235\sqrt{P_{40}}$	$0.16\sqrt{P_{40}}$	$0.13\sqrt{P_{40}}$
δ_{me}	80	$6.5\ P_{80}$	$4\ P_{80}$	$2\ P_{80}$
$\delta_{me.\ sq}$	80	$0.424\sqrt{P_{80}}$	$0.316\sqrt{P_{80}}$	$0.177\sqrt{P_{80}}$
δ_{me}	140	$13.1\ P_{140}$	$10\ P_{140}$	$5.75\ P_{140}$
$\delta_{me.\ sq}$	140	$0.8\sqrt{P_{140}}$	$0.5\sqrt{P_{140}}$	$0.27\sqrt{P_{140}}$

flowing from Kotel'nikov's theorem, is always met in practice. Thus, the zero (null) frequency is determined by the indicating instrument's dynamics.

The practical behavior of a pointer indicating electromagnetic galvanometer, measuring mean current of a condenser (or transformer) receiver, becomes completely satisfactory for frequencies at the receiver input of 4 to 5 cps (current pulses of frequencies from 8 to 10 cps flow through the instrument).

The speed of action of a telemetry device is defined as both the time to establish the frequency at the transmitter's output and as the time necessary to transform the frequency to a new indication of the indicating device (for analog reproduction) or, again, as the time necessary to translate the frequency to code (for digital reproduction). In the very fastest transmitter, the maximum time for establishing the output frequency will be when the signal changes from its nominal value to its null value. This time is defined by the formula

$$T_1 \leqslant \frac{1}{2f_0}. \quad (13)$$

The speed of a telemetering system with analog reception is defined as the time to establish the indication of the indicating instrument and, for a frequency range of 4 to 20 cps, is about 4 seconds.

For digital reproduction, the time necessary to translate the frequencies into code is determined by the receiver error β_{te} and by the frequencies corresponding to the zero value and the nominal value of the measured parameter.

With simple pulse counting,

$$T_2 = \frac{100}{\beta_{te}\%\,(f_N - f_0)}. \quad (14)$$

With a doubled frequency and a transformation of PFM II to PFM I (pulse-frequency modulation with constant pulse duration),

$$T_2 = \frac{50}{\beta_{te}\%\,(f_N - f_0)}. \quad (15)$$

4. Secondary Channel Bandwidth

As was mentioned earlier, when signals are transmitted by PFM II, the subcarrier frequency filter must pass at least the third harmonic. However, it is possible to use transmitting and receiving devices of a pulse-frequency telemetry system with narrow-band filter, passing only the first harmonic of the nominal frequency. With this, teletransmission in a large part of the frequency range will be implemented, not by PFM II, but by FM. In this case, thanks to the narrow-band filters, the signal-to-noise ratio at the receiver input will be reduced.

In the devices with secondary compression used today, the narrowest secondary channel is a subchannel of an 80 cps width. Since the receiving device of pulse-frequency telemetry provides normal operation when a sinusoidal signal whose frequency is of the order of 20 cps is applied to its input, telemetry devices with such stations will operate reliably.

Summary

1. Decreasing the f_0/f_N ratio lowers the requirements placed on the accuracy of transforming dc (voltage) to frequency and on the inverse transformation of frequency to current.

2. Increasing the frequency range, f_N to f_0, for the same filter passband, first decreases the mean square error for weak and for strong noise and, second, decreases the conversion (coding) time when digital reproduction is used.

3. The highest teletransmission frequency for PFM II is determined by the noise from the 50 cps line and by the existing standard telegraph filters. For filters with a 140 cps passband, it is desirable to transmit the signal's third harmonic. The highest frequency is about 20 cps.

4. The lowest (null) frequency with analog reproduction is determined by the indicating instrument's dynamics, and is about 4 to 5 cps.

5. For a nominal teletransmission frequency of the order of 20 cps, a telemetering device can operate with the use of any telegraphic filter.

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*See English translation.

ON THE RELIABILITY OF SCHEMES FOR CONNECTING DISPERSED OBJECTS

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The paper considers the reliability of schemes for the parallel and series connection of a remote control system's executive points in a communications line. The domain of applicability for each of these schemes is shown.

In [1], the reliability of various remote control line structures for dispersed objects was considered. As a development of this work, we consider here the reliability of schemes for connecting the executive points in communications lines. The choice of some connection scheme or another determines, to a significant degree, the structure, and significantly affects the structural reliability, of remote control systems with dispersed objects (oil and gas industries, pipelines, irrigation systems, transportation, mines and factories, cities' communal economies, etc.). To a certain extent, this also applies to automated systems.

Figures 1, 2, and 3 show schemes for parallel, series, and mixed connections of a group of m dispersed executive points (EP) to a common communications bus via distributive devices D which might be parts of the EP's. In a scheme of the mixed type, the individual distributive devices implement, for example, only the functions of switching devices (Fig. 3). Analogous schemes are also used for radio channels.

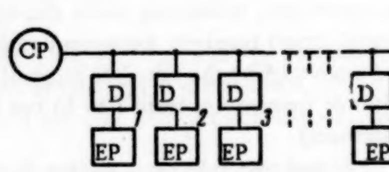

Fig. 1

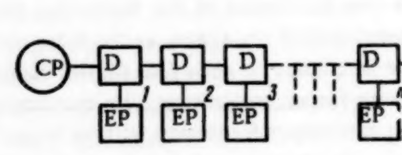

Fig. 2

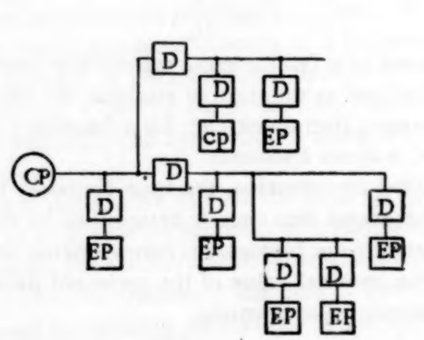

Fig. 3

The executive points on Fig. 1 are connected in parallel via the distributive devices D to a common communications line which joins them to the central point (CP). In many cases, the distributive devices are executed in the form of matching transformers. The parallel connection scheme (Fig. 1) is employed in remote control systems with either frequency or temporal separation of the signals.

With series connections (Fig. 2), the communications line is divided into a chain of segments which are connected in series via the distributive devices D.

In one of the simplest remote control systems with dispersed switching [2], each distributive device contains relay contacts, connected in series in the line, and disconnecting the remaining parts of the system in one of the relay positions. When a series of selection pulses is transmitted, the "next" link is connected to the line at each cycle, and the line is "selected" successively in time.

In remote control systems with dispersed objects, if these systems operate in accordance with the scheme of Fig. 2 and are constructed of contactless elements with rectangular hysteresis loops, the distributive devices are executed in the form of transformers.

With a series connection, each distributive device can also implement the function of signal amplification, similar to what takes place in radio relay lines with the same connection scheme.

We now consider the reliability of systems with parallel and with series connections of the EP's in the communications line. Such connection schemes are the fundamental ones.

We make the following assumptions.

1. All distributive devices in one and the same system (scheme) are identical. For each of them there exists a probability of a short-circuited input or output, P_{sc}, and a probability P_{opc} of an opened circuit, for example, from

a winding blowout. The probabilities P_{sc} and P_{opc} for a given EP are mutually independent, and do not depend on the probabilities of short circuits or open circuits of the other EP's.

The probabilities P_{sc} and P_{opc} of a nonoperative state of a given distributive device are related to the mean frequencies of failure due to short circuits f_{sc} and due to open circuits f_{opc} by the following expressions:

$$\bar{f}_{sc} = \frac{P_{sc}}{\bar{T}_{sc}} \text{ and } \bar{f}_{opc} = \frac{P_{opc}}{\bar{T}_{opc}}.$$

Here, $\bar{T}_{sc}$ and $\bar{T}_{opc}$ are the mean nonoperating times of a given distributive device due, respectively, to short circuits and to open circuits during the period of time between maintenance cycles;

$$\bar{T}_{sc} = \frac{1}{n} \lim_{n\to\infty} \sum_{i=1}^{n} T_{i\,sc}, \quad \bar{T}_{opc} = \frac{1}{n} \lim_{n\to\infty} \sum_{i=1}^{n} T_{i\,opc}$$

where n is the number of faults (failures).

2. The communications line itself is considered to be absolutely reliable. Such an assumption simplifies the analysis and can not change the character of the results in any essential way since, when account is taken of the probabilities of short circuits and open circuits in identical links of the line, there is only an increase in the corresponding P_{sc} and P_{opc}.

Then, for schemes with parallel and with series connections, the probability of system failure due to short circuiting of the distributive devices to ground or to the second conductor in a two-conductor line, without taking account of failures due to open circuits, will be expressed by the formula

$$P = 1 - (1 - P_{sc})^m$$

For $mP_{sc} \ll 1$,

$$P \approx mP_{sc}.$$

The probability of a nonoperative state of an object in a system due to open circuits in the distributive devices when parallel connections are employed (Fig. 1) is $P = P_{opc}$, since the open circuiting of any distributive device disrupts the operation only of the EP connected to it.

For a scheme with series connections (Fig. 2), the mean probability of a nonoperative state of an object in the system due to open circuits in the distributive devices, without account being taken of faults due to short circuits, will be

$$P = 1 - \frac{1}{m} \sum_{i=1}^{m} (1 - P_{opc})^{n_i},$$

where n_i is the number of objects connected between the CP and the ith distributive device ($n_i = i$) and, for $mP_{opc} \ll 1$, will be

$$P = \frac{m+1}{2} P_{opc}$$

If an open circuit in a distributive device disrupts only the operation of the EP connected to it in a parallel-connected scheme, the same open circuit in a series-connected scheme will cause failure of an average of $(m+1)/2$ EP's.

It also follows from the expressions obtained that the probabilities of an object's nonoperative state due to a short circuit in a distributive device in systems with parallel and with series connections are equal to each other when the probabilities of short-circuited distributive devices are equal. However, in a parallel-connected scheme, the probability of a nonoperative state of the system due to a short circuit in the distributive devices or in the EP's can be decreased to an arbitrarily small quantity by connecting the EP's to the line via cutouts or limiters similarly to what is done in connecting electric energy users to a power line, or radio relay points to a radio relay network.

For frequency systems of remote control, and for systems which use exponential transformers [3] for the time separation of signals, the EP apparatus is comparatively simple in parallel-connected schemes.

If existing systems of remote control with time separation of signals are connected in accordance with the scheme of Fig. 1, it is necessary to have, at each EP, a one-terminal set (incomplete in many cases) which is frequently unreasonable (for a small number of signals on one EP).

The series-connected scheme can be realized in remote control systems with time separation of signals with contact distributed switches [2] or in contactless systems with elements with rectangular hysteresis loops.

Such systems, even when constructed from contactless elements, have significantly less structural reliability, as our analysis shows.

Frequency systems of remote control can also be constructed in accordance with the series-connected scheme (Fig. 2), but this is unreasonable, in connection with the lower structural reliability of such systems.

The use of series-connected schemes, from the point of view of reliability, may be justified principally in those cases when it is necessary to amplify signals at given points of the line, or it is necessary to switch groups of EP's in communications lines with tree structures, if the solution of such problems gives rise to difficulties when parallel-connected schemes are used.

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*See English translation.

ON THE QUESTION OF RELAY-CONTACT DEVICE RELIABILITY

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Methods are considered for increasing the reliability of operation of relay-contact devices by means of spare elements. A new method is suggested for sequential circuits which allows the device's reliability to be increased with the minimum number of spare relays.

In recent years the questions of reliability of electrical devices have received great attention. With this, the problem of increasing reliability is solved by two routes. The first of these is the increase of strength and durability of the elements entering into devices, the replacement of contact elements by contactless ones, for example, by semiconducting and magnetic elements, etc. The second route, independent of the first, is the wide use of stand-by elements, assemblies and blocks, i.e., the use of back-up.

In these cases, the operating parts of the apparatus or of its individual blocks, or even individual elements, are supplied with reserves which are either switched into operation automatically when the basic set fails or operate constantly in parallel with it. Such reserve elements are "redundant," since their introduction adds no new functions to the device.

As an example of the use of the back-up method of increasing reliability, one may adduce the automatic track blocking circuit of the Moscow subway in which, as is well known, a type DSR relay is connected in parallel for every two tracks.

In the engineering and scientific literature devoted to the questions of stand-by capability, one considers principally the reliability of radioelectronic apparatus, constructed from contactless elements and used in a number of special domains, for example, in computing technology, in radio relay communications, for the control of moving objects, etc. In correspondence with the purpose of such apparatus, there arises the requirement that it operate without failures during some interval of time, so that here reliability is defined as the probability that, in the course of a definite time interval, the device will correctly carry out its function.

In industrial devices for automation and remote control, one ordinarily considers a device to be sufficiently reliable if a random failure of its elements does not entail an incorrect functioning which leads to an unsafe state of the object. Operational failures in these cases are considered as "protective", and are ordinarily accompanied by either a blocking device and an alarm indication of the fault, or a transfer of the device to a safer state. Thus, for example, when a railway element fails in the automatic blocking circuit, a red light signal automatically appears, requiring that the train stop.

However, as is understandable, each such case of protective failure disrupts the continuity of the technological process. With the modern tendency to intensify production, it becomes important to increase the reliability and unceasing operation of industrial automation and remote control devices.

In the present paper we consider several questions in the construction of reliable relay-contact devices which permit a single failure to occur without disruption of operation. The principles to be considered in the sequel may be extended also to contactless devices using relay-acting (switching) elements.

The effect achieved when the ordinary method of parallel back-up [1] is used can be illustrated by the example of the simplest relay-contact circuit (Fig. 1a), whose function is to close circuit a-b when relay A1 is excited.

We denote the probability of correct closure of the relay contact by $p(A) = p$, and the probability of the event that circuit a-b will be closed by $p(a\text{-}b)$. In the given case, $p(a\text{-}b) = p$.

Connecting relay A2 in parallel to relay A1 (simple back-up) and connecting the contacts of relay A1 and A2 in parallel (Fig. 1,b) increases the probability of correct closure of circuit a-b up to the value $p(a\text{-}b) = 2p - p^2$, i.e., for $1 - p \ll 1$, increases the reliability by almost one order.

Conversely, a series connection of the contacts of relays A1 and A2 (Fig. 1c) lowers the probability of closure of circuit a-b to the value $p(a\text{-}b) = p^2$.

Figure 1 d-f, also gives several variants of the simplest schemes for the two- and three-fold back-up of relay A1, and the corresponding values of the probability of closure of circuit a-b. To determine the expressions for the probabilities, it was assumed that all the relays A1, A2, A3, and A4 were monotypic, as were their contacts, that the probability of contact closure was the same for all, and that cases of incorrect operation were statistically independent.

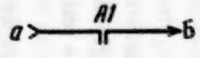
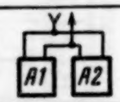
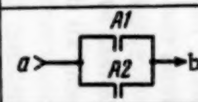
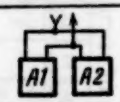
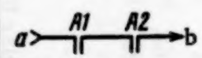
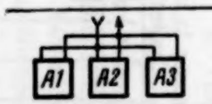
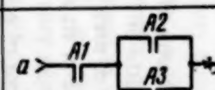
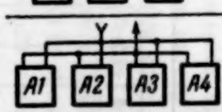
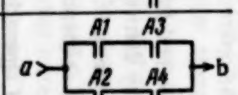
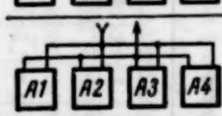
	Relay winding connection	Contact connection	Probability: Closed circuit a-b	Probability: Open circuit a-b
a			$p(a-b) = p$	$q(a-b) = 1-p$
b			$p(a-b) = 1-(1-p)^2 = 2p - p_2$	$q(a-b) = (1-p)^2 = 1-2p+p^2$
c			$p(a-b) = p^2$	$q(a-b) = 1-p^2$
d			$p(a-b) = [1-(1-p)^2]p = 2p^2 - p^3$	$q(a-b) = 1-2p^2+p^3$
e			$p(a-b) = 1-(1-p^2)^2 = 2p^2 - p^4$	$q(a-b) = (1-p^2)^2 = 1-2p^2+p^4$
f			$p(a-b) = [1-(1-p)^2]^2 = 4p^2 - 4p^3 + p^4$	$q(a-b) = 1-[1-(1-p)^2]^2 = 1-4p^2+4p^3-p^4$

Fig. 1.

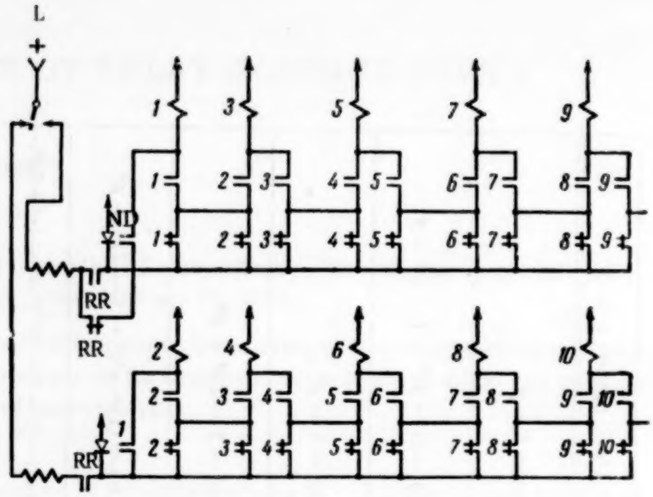


Fig. 2

As is obvious from these examples, by paralleling the corresponding number of relays (with multiple back-up) and by properly connecting their contacts in a circuit which is equivalent in its operation to the action of one contact (or any other more complicated structure), one can attain any previously given degree of reliability, even with the use of elements which are themselves of low reliability. Of course, such back-up leads to a significant increase in complexity and cost of the devices, and to a large expenditure of circuit elements.

For sequential relay-contact circuits, a more economical back-up, which may be called "combined" is possible. The essence of this back-up amounts to this, that the back-up elements, the number of which, as a rule, is less than the number of the basic elements, back each of the basic elements up at that moment (cycle) when the operation of the entire device is determined by the operation of this basic element.

We consider this new back-up method in terms of the example of a centralized control relay distributor of type DVK-3 – DVK-3a. The functional schematic of such a distributor is shown in Fig. 2, and the sequence of its actions is given in Table 1 [2]. One easily convinces oneself that a fault in any one of the counter relays 1-10, or of their contacts, causes the distributor to cease operation, as well as a fault in the action of the entire one-terminal set (and, if this one-terminal set is the central control point, the entire system fails).

We isolate, in Table 1, those cycles which would be terminal points, completing each distributor step, i.e., cycles 0,3,6,9,12,15, etc., and we write the state of the distributor at each of these cycles in the form of a binary number. We adopt the convention that an excited relay state will be denoted by a "1", an unexcited state by a "0". Table 2 gives the sequence of these binary numbers, which completely characterizes the sequence of operations of the distributor relays and which mirrors, by the corresponding binary numbers, their states at each step. Thus, for example, the fifteenth cycle of Table 1 corresponds to the binary number 000001000010 (the fifth cycle of Table 2), while the eighteenth corresponds to 000010000000.

By considering this table of binary numbers as a set defined by given combinations of relay states, and considering all the rest of the set of binary numbers which are not

TABLE 1 Sequence of Operations of a Relay Distributor L is the line relay, RR is the receiving relay.

Relay	Cycles of circuit operation 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	etc.
RR				+																		
L	+			−			+			−			+			−			+			
1		+				−																
2					+				−													
3								+				−										
4											+				−							
5														+				−				
6																	+				−	
7																				+		
8																						
9																						
10																						

TABLE 2 Sequence of Distributor Operations, Given in the Form of a Table of Binary Numbers

Cycles	Circuit elements											
	10	9	8	7	6	5	4	3	2	1	L	RR
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	1	1	1
2	0	0	0	0	0	0	0	0	1	0	0	0
3	0	0	0	0	0	0	0	1	0	0	1	0
4	0	0	0	0	0	0	1	0	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	1	0
6	0	0	0	0	1	0	0	0	0	0	0	0
7	0	0	0	1	0	0	0	0	0	0	1	0
8	0	0	1	0	0	0	0	0	0	0	0	0
9	0	1	0	0	0	0	0	0	0	0	1	0
10	1	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	0

TABLE 3 Distances between Relay State Combinations at Each Cycle of Distributor Operation

Cycle numbers	Cycle numbers										
	0	1	2	3	4	5	6	7	8	9	10
0		3	1	2	1	2	1	2	1	2	1
1	3		4	3	4	3	4	3	4	3	4
2	1	4		3	2	3	2	3	2	3	2
3	2	3	3		3	2	3	2	3	2	3
4	1	4	2	3		3	2	3	2	3	2
5	2	3	3	2	3		3	2	3	2	3
6	4	4	2	3	2	3		3	2	3	2
7	2	3	3	2	3	2	3		3	2	3
8	1	4	2	3	2	3	2	3		3	2
9	2	3	3	2	3	2	3	2	3		3
10	1	4	2	3	2	3	2	3	2	3	

written in the table as distortions of these combinations, one can try to use, for constructing self-correcting circuits, the same methods which were developed for constructing self-correcting codes, also described by binary numbers which are tabulated in analogous fashion [3,4].

It is well known that, for the construction of error-free codes, it is necessary to create redundant elements of the combinations, and thereby to achieve the result that the code combinations differ from each other by a sufficient number of bits ("distance" between combinations). Thus, to construct a code for the correction of unitary errors, it is necessary that the code combinations differ from each other by at least three bits.

We now extend these propositions to the given combination of relay states. Table 3 gives the values of the "distances" between the combinations of relay states (the binary numbers of Table 2) for the distributor of the circuit in Fig. 2, from which it is clear that, in a number of cases, this condition is not satisfied, so that it is required to introduce auxiliary elements in the combinations (relays) to provide the requisite redundancy.

The number of auxiliary relays can be determined from the well-known expression of Hamming,

$$2^{n_0} \leqslant \frac{2^n}{n+1},$$

where n_0 is the number of basic relays in the circuit and $\underline{n}$ is the total number of relays, including the auxiliary ones.

The value of $m = n - n_0$, i.e., the number of auxiliary relays, can be found from the following table, set up in accordance with this formula:

$$\begin{array}{llllllllllll} n_0 = 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ n = 3 & 5 & 6 & 7 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ m = 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{array}$$

In the case under consideration, it turns out to be sufficient to add four auxiliary relays. Connecting these relays in the over-all operation of the distributor can be implemented in various ways with the same condition that repeated combinations not occur. As is well known, the number of nonrepeating combinations with this will equal 2^n, i.e., the number of arrangements which differ from each other in at least one place. It is most convenient to construct the table of auxiliary relay operations such that, at each cycle, the minimum number of auxiliary relays, i.e., one relay, will change state. This condition is satisfied by, for example, the Gray code [5], used in a somewhat different form for constructing Table 4, giving the operation of auxiliary relays I-IV. One easily convinces oneself that, with this, the distance between the binary numbers of the combinations of relay states at different cycles of distributor operation in no case is less than three, which is the requirement imposed.

We now consider three cases of constructing self-correcting circuits for relay counters.

1. Circuits for self-correction it there appears, in a binary number characterizing an operate cycle of a given relay, a zero instead of a one (for example, if a prior counter relay did not close its contact in the same operate circuit of the given relay).

2. The same, but for the appearance of a one in the binary number instead of a zero (for example, for the sealing of a contact of the prior relay).

3. The same, but for the appearance in a binary number of a zero instead of a one, or a one instead of a zero.

In the first case, to provide for operation of the given next counter relay, its operate circuit, passing through normally open contacts of the previous counter relay, must be backed up by the contacts of that auxiliary relay which also changed its state at the previous cycle (a normally open contact if it were excited, and a normally closed contact if it were unblocked).

In the second case, protection from a false operation on an incorrect cycle due to a random closing of a contact of a previous relay is attained by introducing into the operate circuit of the given relay the contacts of three auxiliary (back-up) relays which do not change their state at the given cycle, and the combination of whose states at the given cycle differs by the state of at least one relay from

TABLE 4 Binary Numbers for Operation of the Distributor with Auxiliary Relays

Cycles	Basic relays of the distribution circuit												Auxiliary relays				Back-up circuit contacts			
	10	9	8	7	6	5	4	3	2	1	Л	ПЛ	IV	III	II	I	IV	III	II	I
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
1	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	NC	NC	NC	—
2	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	NC	NC	—	NO
3	0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	NC	NC	NO	—
4	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	NC	—	NO	NC
5	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	NC	NO	—	NC
6	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	—	NO	NC	NC
7	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	NO	—	NC	NC
8	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	NO	NC	—	NC
9	0	1	0	0	0	0	0	0	0	0	1	0	1	0	1	1	NO	NC	NO	—
10	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	NO	NC	—	NO

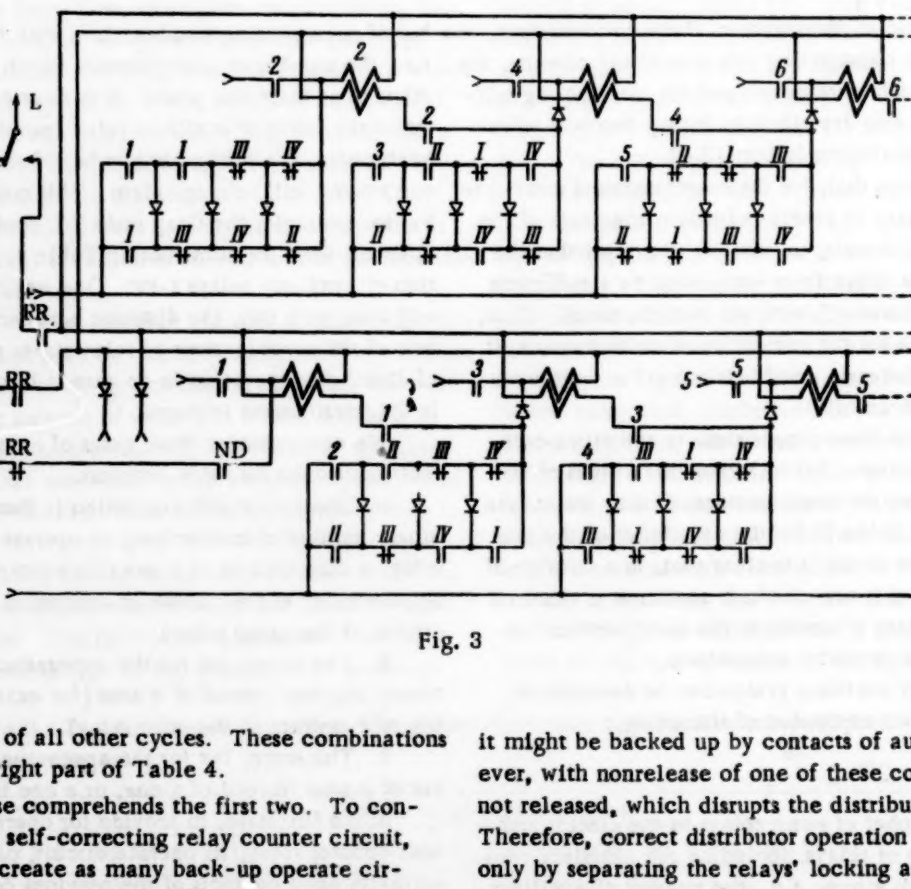


Fig. 3

the combinations of all other cycles. These combinations are given in the right part of Table 4.

The third case comprehends the first two. To construct a protected self-correcting relay counter circuit, it is necessary to create as many back-up operate circuits as there are series contacts of basic and auxiliary relays in the basic operate circuits. Certainly, the contacts of these back-up circuits can be joined by the use of diode elements, as shown in Fig. 3.

It is necessary to dwell on the cut-off relay circuits. As is clear from the scheme of Fig. 2, the blocking and cut-off circuits of the counter relays of an ordinary relay distributor are interconnected, and pass in series through inherently normally open contacts and normally closed contacts of the relay of the next higher ordinal number. For the case of nonclosure of this normally closed contact, it might be backed up by contacts of auxiliary relays. However, with nonrelease of one of these contacts, the relay is not released, which disrupts the distributor's operation. Therefore, correct distributor operation can be achieved only by separating the relays' locking and cut-off circuits, whereby the release of the relays is implemented, not by means of the supply circuit, but by counter flows in the relays' second windings (active unlocking). With this method, unlocking of a given counter relay is obtained by means of the operate circuit of the next relay of the same parity (i.e., even or odd, respectively), and may therefore be identified with it, the differentiation being only by the state of the forcing element, and the cycles on which they are used.

Figure 3 shows the circuit for connecting several of the basic counter relays of a self-correcting distributor, con-

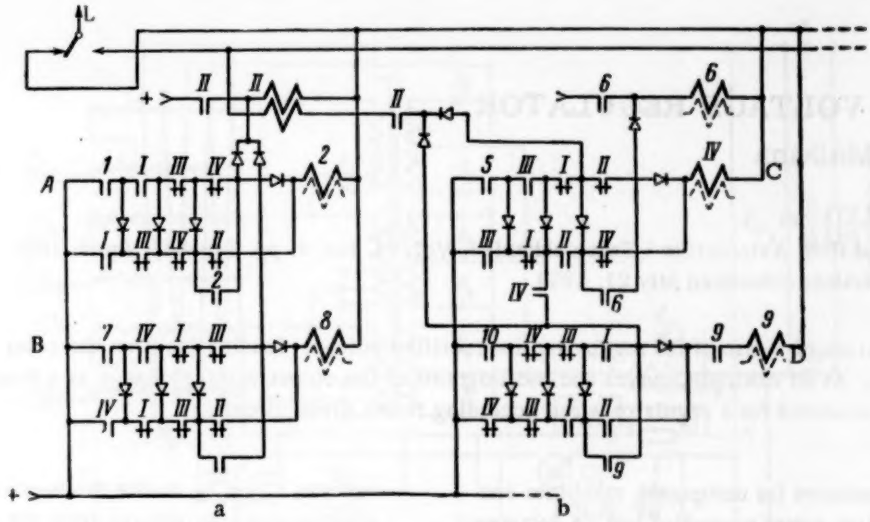


Fig. 4. a is the operate circuit of relay II, b is the release circuit of relay II; A is the operate circuit of relay 2, B is the operate circuit of relay 8, C is the operate circuit of relays 6 and IV, D is the release circuit of relay 9.

structed in accordance with Table 4 and with the condi-tions to be given now.

As was stated earlier, auxiliary relays I-IV are connected analogously to the basic counter relays in that, at the corresponding cycles of distributor operation, they operate or are released simultaneously with the basic relays. Since auxiliary relays I and II operate and release twice during an operate cycle of the distributor, one then uses for them the binary number of such circuits, as is shown in Fig. 4 for auxiliary relay II.

The difference in the circuits for the basic and the auxiliary relays stems from the fact that there is used, in one of the back-up circuits for operating a basic relay, a proper contact of that auxiliary relay which must also operate (or release) on the same cycle. To obtain coherent operation both of the basic and of the auxiliary relay by this back-up circuit, this circuit is so separated for the basic and the auxiliary relay that the winding of the auxiliary relay is supplied only after the basic relay has operated.

Such a relay distributor provides correct operation for any single fault, i.e., for the failure of any one element, either basic or auxiliary. The distributor's output circuit must, of course, be constructed like an operate circuit, i.e., it must be backed up.

Thus, increasing by almost an order the reliability of distributor operation is attained most economically by adding four relays only, instead of the ten necessary for parallel back-up. It should be mentioned that, according to Hamming's formula, four auxiliary relays can provide back-up for 15 basic relays. *

Using the same principle, one may construct multiple relay contact circuits in which are corrected double, triple, etc. errors, i.e., which will provide correct operation for simultaneous failures of two, three, etc. relays, at the expense of a corresponding increase in redundancy.

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* This last sentence is factually (and mathematically) incorrect, as are the data of Table 3 on page 369. The maximum number of basic relays for which four auxiliary relays will suffice is not 15, but 11 [Publisher's note].

RECTIFIED VOLTAGE REGULATOR

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A new variation of a circuit for regulating low rectified-voltage feeding loads on the order of hundreds of watts is discussed. As an example, curves and oscillograms of the output voltage change as a function of the load change are presented for a regulator wired according to the given circuit.

The use of transistors for computing machines and other devices requires power sources of low dc voltages. These voltages may be obtained by lowering and rectifying the ac power line voltage. The use of ac sources means that filters and voltage regulators must be built. The ac source can be ether single- or three-phase.

One of the advantages of using rectified voltages from three-phase ac sources is that the ripple on the output is less; this simplifies and eases the filter requirements of the voltage regulator. The voltage regulators considered in the article are designed for loads on the order of hundreds of watts, and use saturable reactors for regulation. These regulators permit ac line voltage fluctuations of ± 5-10%, a temperature change of the surrounding medium from -30 to +50°C, and continuous control of the regulated voltage over a range of ± 20-25%.

The fluctuations of the regulated voltage are less than ±0.5-1.5% for this type of regulator, and the full regulation time for a step change of load from 100% to 50% and, conversely, from 50% to 100% of the nominal value does not exceed 5-15 msec.

The circuit diagram of the regulator is shown in Fig. 1.

The regulator consists of the following basic elements:

1. Saturable reactors $\overline{Ch}_1$ - $\overline{Ch}_3$, made with an external feedback winding W_0 and one control winding W_y, shunted by the capacitance C_y in order to eliminate the effect of the inductive emf of even harmonics. The reactors can also be made with an internal feedback winding.

2. The three-phase transformer Tr (or three single-phase transformers).

3. Power rectifiers B_1-B_6.

4. The filter, consisting of choke $\overline{Ch}_f$ and capacitances C_{f_1} and C_{f_2}.

5. The meter-amplifier element which, in its turn, consists of: a) voltage divider made of resistances R_1-R_3, and R_2-R_4; b) Zener reference diodes V_{c_1} and V_{c_2}, type D809 or D810; c) rectifiers B_7-B_{10}, type D7D-D7Zh; d) junction transistors Q_1, Q_2, and Q_3 (type P4B); e) resistances R_5, R_6, and R_7, which limit the currents in the control winding and transistors Q_1 and Q_2.

The ac of the three-phase power line, at a 400 cps frequency, is connected through three single-phase saturable reactors $\overline{Ch}_1$-$\overline{Ch}_3$, to the three-phase transformer. The stepped-down ac voltage from the secondary windings of the transformer is rectified by rectifiers B_1-B_6, the variable component is filtered out by the filter ($\overline{Ch}_f$, C_{f_1} and C_{f_2}), and the resultant dc is fed to the load.

In this case, the saturable reactors operate as single-phase magnetic amplifiers, working into a variable load resistance. A simplified equivalent circuit diagram of the load windings of such an amplifier is shown in Fig. 2a.

Commercial transformers have an insignificant leakage reactance, and the quantities x_1 and x'_2 can be neglected. If the load resistance is large, it is also possible to neglect the choke resistance (r_{ch}), and primary and secondary winding resistance (r_1 and r'_2) and the core loss resistance (r_0). Therefore, the simplified equivalent circuit diagram of the load windings of the magnetic amplifier corresponds to the circuit shown in Fig. 2b, for no-load on the regulator.

The amplitude of the output voltage U_{out} will be determined, in this case, by the relation between the choke (magnetic amplifier) reactance x_{ch}, and the transformer reactance x_0. The relation chosen between x_{ch} and x_0 and the given value of the regulated voltage determine the transformation ratio of the transformer.

When the transformer is loaded, $R_H^1 \ll x_0$, and the simplified equivalent circuit of the load windings of the saturable reactor corresponds to the circuit shown in Fig. 2c.

The regulation of the output voltage in this type of regulator is based on the fact that the reactance of the saturable reactor x_{ch} depends on the magnitude of the dc excitation ampere-turns.

The main excitation of the circuit is caused by the load current, that is, by the ampere-turns of the feedback winding, which has a coefficient somewhat less than unity (this coefficient can even be equal to unity). The load current is minimum at no-load, and is determined by the currents of the voltage dividers R_1+R_3 and R_2+R_4, and the resistance x_{ch} is maximum (in the circuit of Fig. 2b).

At maximum load, x_{ch} is minimum (in the circuit of Fig. 2c), and the change in x_{ch} due to load current compensates for the change in the output voltage due to the change in the voltage drop across the circuit resistance $R = r_{ch} + r_1 + r_2$.

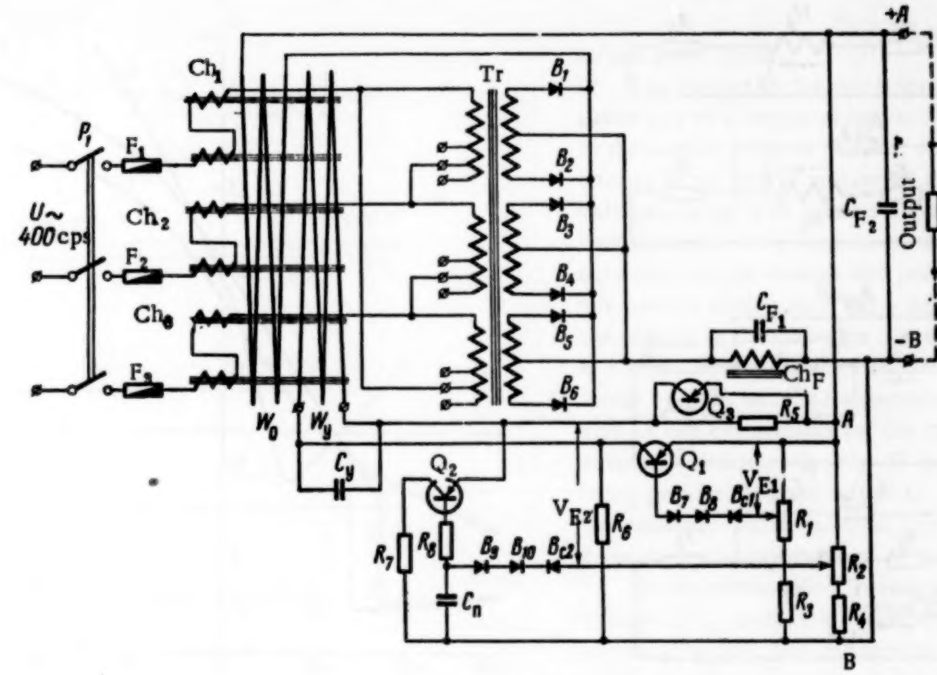

Fig. 1

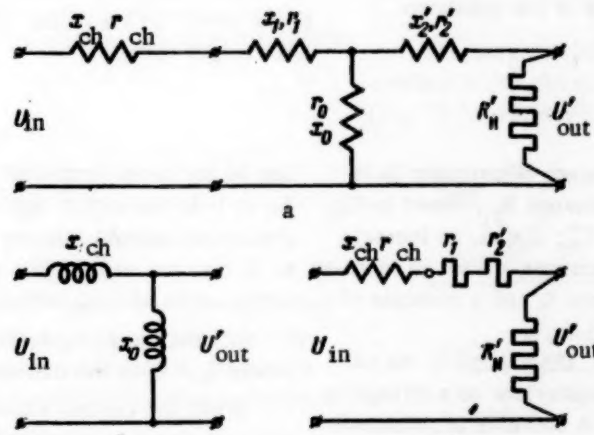

Fig. 2. Simplified equivalent circuit diagram of the load windings of a single-phase saturable reactor at different loads.

The dc excitation of the saturable reactors is caused not only by the ampere-turns of the feedback winding W_0, but also by the ampere-turns of the control winding W_y. The latter is used to get better stabilization of the output voltage, and to control the amplitude of the regulated voltage over a range of ± 20-25%.

The direction and amplitude of the current in the control winding depend on the states of transistors Q_1 and Q_2 in the metering element.

Voltage reference diodes (B_{c_1} and B_{c_2}) are wired in the base circuit of each transistor (Q_1 and Q_2). Therefore, the circuit in the transistor base will, in practice, flow only when the voltage between the moving arm of the divider and the emitter of the given transistor (U_{e_1} or U_{e_2}) is greater than the sum voltage of the reference diode (U_{r_1} or U_{r_2}) and the rectifiers ($U_{v(7-8)}$ or $U_{v(9-10)}$). The greater the difference in the voltages mentioned, the greater the base current of the transistor will be, and the greater the collector current.

When both transistors are cut off, the circuit diagram of the control winding will be practically analogous to the circuit shown in Fig. 3a. The ampere-turns of the control winding in this case assist the excitation action of the feedback winding, and by the same token, cause additional decrease of the reactance x_{ch}.

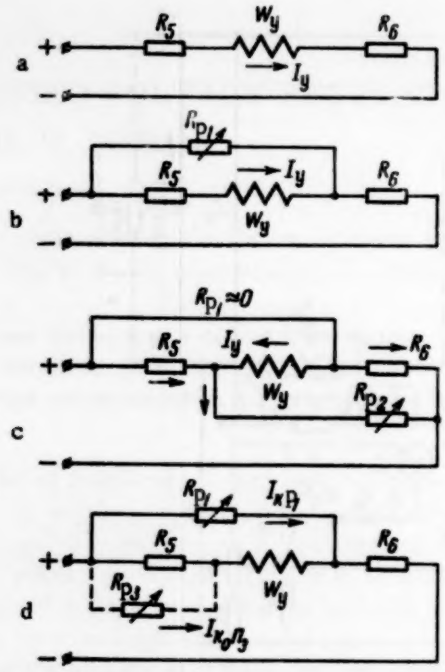

Fig. 3. Simplified equivalent-circuit diagrams of the control winding of the saturable reactor for different states of the transistors in the metering element.

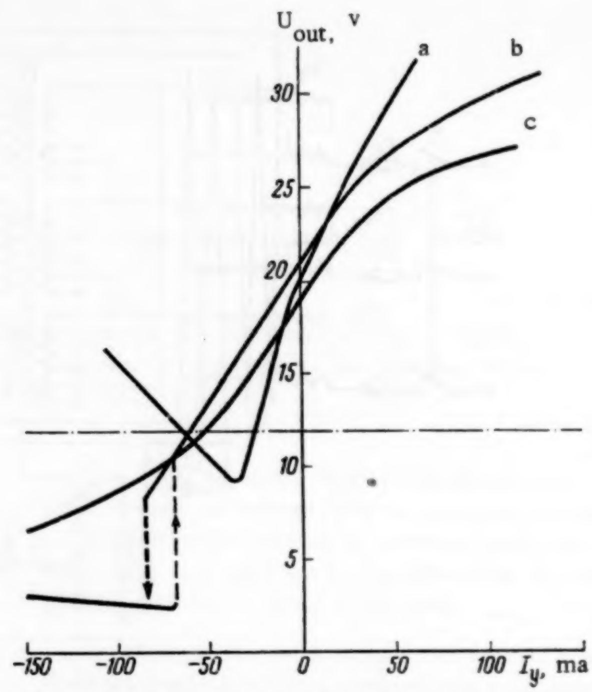

Fig. 4. Dependence of the output voltage of the regulator on the current in the control winding. a) for R_H = 30 ohms, b) 4.2 ohms, c) 1.2 ohms, d) minimum value of the regulator output voltage.

A change in the collector current of transistor Q_1 is equivalent to a change in the resistance R_{p_1}, shown in Fig. 3b. A decrease in the resistance R_{p_1}, that is, an increase in the collector current of this transistor, causes an increase in the voltage drop across resistance R_6 and a decrease of the current in the control winding.

When transistor Q_1 is fully on, the change in the collector current in transistor Q_2 is equivalent to a change in resistance R_{p_2}, shown in Fig. 3c. A decrease in resistance R_{p_2} will cause an increase of an opposing current in the control winding, and therefore these ampere-turns will cause a relative increase in the reactance x_{ch}.

The use of feedback with a coefficient approaching unity ensures an abrupt change in the reactance x_{ch}, and consequently, in the output voltage, when the current in the control winding is changed.

The change in the output voltage of the regulator as the current in the control winding is changed is shown in Fig. 4; the control winding of this regulator has been disconnected from the metering element, and connected to an independent source of dc.

It can be seen from Fig. 4 that to change the output voltage of the given regulator from 12 to 25 v across a load resistance of 30 ohms, it is sufficient to change the control current from -25 to +25 ma, and from -50 to +50 ma for small load resistances. There are 70 turns on the control winding of the reactors of the given regulator. It can also be seen from Fig. 4 that a negative control current exceeding 50 ma is not permitted in this type of regulator. In order to hold the output regulated voltage at 25 v, for example, the control current must be altered within the limits of 22 to 52 ma, and to hold this voltage at 20 v, the control current must be altered between -6 to +8 ma.

All the foregoing determines the choice of the resistances R_5 - R_7 in the metering element.

When the control winding is connected to the metering element, the output voltage is stabilized because a change in the output voltage due to a variation in the load, or power line voltage, causes the base current of a transistor to change; this, in turn, alters the control current so as to counteract the output voltage change.

In principle, it is not possible to compensate completely for the voltage deviation by changing the current in the control winding, since there must always be some voltage difference to cause a new value of the control current. However, the output voltage is maintained quite accurately due to the large amplification factor.

Some curves of the dependence of the output voltage on the load change are shown in Fig. 5 for one of the regulators of the type discussed in this paper. These dependencies are given for different ac source voltages.

The amplitude of the regulated output voltage of this type of regulator is determined by the position of the moving arms of the voltage dividers, which consist of resistances R_1, R_3, and R_2, R_4.

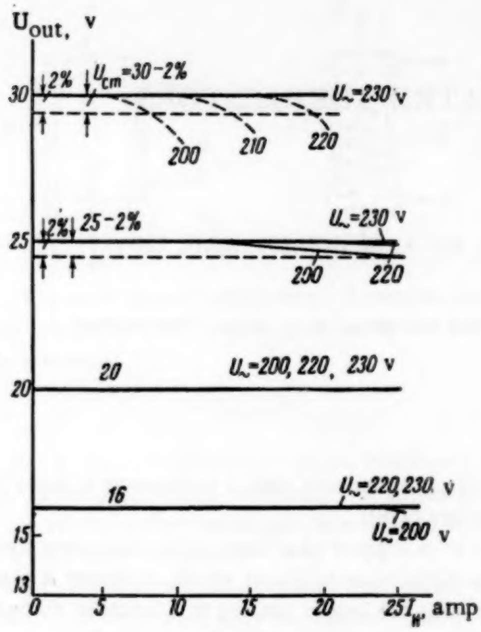

Fig. 5.

The currents in the dividers are significantly greater than the base currents of the transistors; therefore the dividers are only slightly loaded.

The voltage obeys the relation

$$U_{out} = U_{E1}\frac{R_1 + R_3}{R_{x'}},$$

for the first divider, where R_x is the resistance of the divider from point A to the moving arm.

Since the voltage U_{e_1} can be greater under regulation than the sum of the voltages $U_{r_1} + U_{V(7-8)}$ by only a very small amount, an increase in the resistance R_x (that is, a movement of the arm of the divider) will cause a decrease in the output voltage amplitude

The regulated output voltage can be lowered by moving the arm of the first divider from point A to point B, but only to a value which does not require a change in the direction of the current flow in the control winding. Further reduction of the output voltage can be realized by moving the arms of both voltage dividers. For finer control, regulation of the output voltage must be accomplished by moving the arms and the voltage dividers simultaneously, so that the resistance ratios of the dividers

$$\frac{R_1 + R_3}{R_x} \text{ and } \frac{R_2 + R_4}{R_{x_2}}$$

do not differ much.

The change in the collector currents of junction transistors due to a temperature variation leads to a decrease in the control winding current, which in turn causes a change in the output voltage of the regulator. The output voltage change will cause a corresponding variation in the base currents of the transistors, which compensates for the collector current change and practically restores the output voltage amplitude. But when the regulator is heated, the change in the collector current of the transistor due to a temperature increase must be less than the initial collector current, so that the decrease in the transistor base current can compensate for the increase in the collector current. Transistor Q_3 (Fig. 1) is used to prevent ΔI_c from being greater than the initial I_c. With no base connection, the collector current of Q_3 is minimum, and its increase depends on the temperature of the environment.

As the temperature increases, the collector current change of Q_1 leads to a decrease in the control winding current (Fig. 3d), but an increase in the collector current of Q_3, conversely, causes an increase in the control winding current. Thus, the collector current change in Q_3 in this case compensates for the change in the collector current of Q_1.

The output voltage change for a regulator wired according to the circuit discussed is shown on oscillograms (Fig. 6) for various load changes.

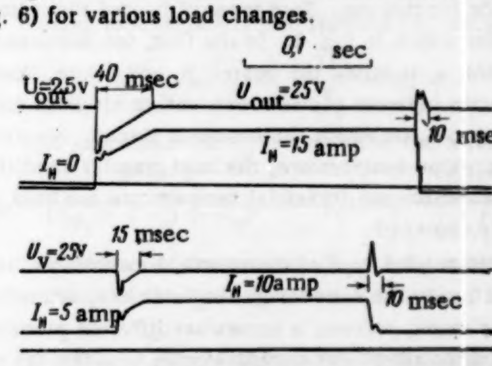

Fig. 6. a) Output voltage change for load current changes from zero to 15 amp and from 15 amp to zero, b) output voltage change for a load current change from 5 to 10 amp, and from 10 to 5 amp.

The recovery time obtained for this particular regulator is not a limit for this type of regulator.

STABILIZING THE TEMPERATURE OF HEATED THERMISTORS

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An analytic method of calculating circuits for stabilizing the temperature of heated thermistors used in time delay apparatus is presented.

At present, a number of methods of obtaining time delay with electrical circuits are used. Circuits using the thermal inertia of thermistors are convenient due to their simplicity, reliability, and ability to give long time delays reaching dozens of minutes. However, these circuits are sensitive to a temperature change of the surrounding medium; this limits their use.

In order to eliminate this dependence on the external temperature, Udalov [1] proposed that heated thermistors be used, whose temperature was regulated in a simple manner by changing the current of the heater.

It must be noted that not all heated thermistors are suitable for this use. Two types of heated thermistors exist; they are shown in Fig. 1. In the first, the semiconducting element, a, is inside the heater, b, and hence, heat transfer occurs between the semiconducting element and the envelope, c, on which the heater is placed. By stabilizing the envelope temperature, the heat transfer conditions of the thermistor and its initial temperature are held approximately constant.

The second type of thermistor, in which the heater is placed inside the semiconducting element, which has a tubular shape, presents a somewhat different picture. In this construction, heat transfer occurs between the semiconducting element and the medium; a temperature change of the latter cannot be compensated for by changing the heater current with the aid of the proposed circuit. Only the temperature of the thermistor can be stabilized. Analysis shows that this condition is not sufficient to maintain the delay transient constant when a circuit with the thermistor is turned on, if the temperature of the surrounding medium changes. Consequently, the second type of thermistor will not do for the circuit being considered.

The operation of the circuit can also be made independent from the temperature of the medium by means of thermostation of the thermistor. However, this method is applicable only in those cases where the thermostat is simple in construction. A heated thermistor of the first type can be considered as a microthermostat, inside of which the temperature can be stabilized within certain limits. By analogy, it is possible to make a small-sized thermostat, and place a direct-heated thermistor in it. The possibility of making such a thermostat is mentioned in the literature [2,3].

Let us suppose that there is an insulating cylinder on whose surface, or inside of which, a heater is placed. By regulating the heater current in a suitable manner, the temperature inside the cylinder can be held constant when the medium temperature changes within certain limits. It should be noted that the temperature inside the thermostat must be higher than the temperature of the surrounding medium.

Two satisfactorily simple circuits for controlling the temperature can be proposed; these are shown in Figs. 2 and 3, in which the thermistor acts as the sensing element. In these circuits, R_T is the thermistor in the regulator circuit, $\underline{r}$ is a linear resistance, r_H is the heater resistance, R_{TC} is the semiconducting element of the heated thermistor, and r_b is the barretter. The circuit using the barretter has the advantage over the circuit of Fig. 2, in that it is independent of supply voltage fluctuations, but such a circuit requires somewhat greater power than the circuit with the linear resistance.

The circuit works in the following manner. At a low temperature of the medium, the thermistor resistance has a considerable magnitude, the current in it is small, and a large amount of power is liberated in the heater. As the temperature of the medium is raised, the current in R_T increases, and by the same token, the current in, and power liberated by the heater decrease. Consequently, it is essential to choose the parameters of the circuit so that as the temperature of the medium changes, the temperature inside the thermostat or heated thermistor will remain within given limits.

It has been experimentally established that the heat transfer between the thermistor or thermostat and the surrounding medium follows Newton's formula well within the operating temperature range

$$P = F\alpha (T_T - T_0) = H (T_T - T_0) = K(T - T_0), \tag{1}$$

where P is the power dissipated by the thermistor or thermostat into the surrounding medium, F is the area of the external surface of the thermistor or thermostat, α is the surface heat emission coefficient, H is the overall dissipation factor of the thermistor, T_T and T_0 are the temperatures of the

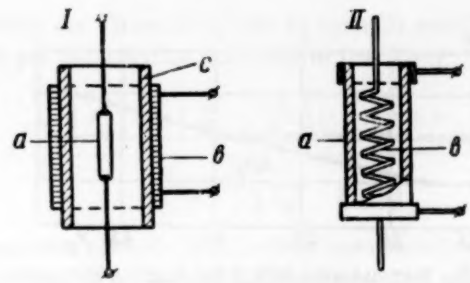

Fig. 1. Types of heated thermistors. I) semiconducting element inside the heater, II) heater inside the semiconducting element.

thermostat surface or the surface of the thermistor heater, and that of the surrounding medium, T is the temperature inside the thermostat or thermistor heater, and K is the over-all heat transfer constant.

Temperature T will be constant if the power dissipated by the resistance r_H is equal to the power emitted by the thermistor or thermostat into the surrounding medium:

$$P_H = I_H^2 r_H = K(T - T_0). \tag{2}$$

If the semiconducting or thermostated element liberates a certain power, it is considered to be a deduction from the computed value of P_H when calculations are made. In time relay circuits, the semiconducting element is ordinarily unexcited before the main circuit is turned on, and thus it is not necessary to account for this power.

Consequently, the power P_H should be a linear function of the medium temperature T_0. The proposed circuits permit one to obtain a dependence $P_H(T_0)$ which is nearly linear in the static mode. In order to have this, it is essential that the curve $P_H(T_0)$ assume values at a number of points in the temperature range which correspond to a linear dependence of $P(T_0)$, that is, so that the difference $\Delta P = P(T_0) - P_H(T_0)$ is zero at these points.

Let us take three such points, which correspond to the limits of the temperature range and its center, as shown in Fig. 4. The curve $P_H(T_0)$ must have an inflection point in order to intersect $P_0(T_0)$ at points 1, 2, and n. This means that d^2P_H/dT_0^2 must be zero at the given point n. Let us consider this condition for the proposed circuit.

For the circuit of Fig. 2, the power liberated in r_H, is determined by the expression

$$P_H = U^2 r_H \frac{R_T^2}{[R_T(r + r_H) + rr_H]^2}, \tag{3}$$

where U is the power supply voltage of the circuit.

Then

$$\frac{d^2P_H}{dT_0^2} = 2U^2 rr_H \tag{4}$$

$$\left\{\frac{[(R_T')^2 + R_T R_T''][R_T(r + r_H) + rr_H] - 3(r + r_H) R_T (R_T')^2}{[R_T(r + r_H) + rr_H]^4}\right\},$$

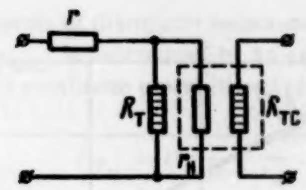

Fig. 2. Circuit for controlling the temperature of a heated thermistor.

whence,

$$\frac{rr_H}{r + r_H} = \frac{R_T[2(R_T')^2 - R_T'' R_T]}{(R_T')^2 + R_T'' R_T}. \tag{5}$$

The temperature dependence of the thermistor is defined by the expression

$$R_T = R_{T_H} \exp\left(\frac{B}{T} - \frac{B}{T_H}\right) = \gamma R_{T_H}, \tag{6}$$

where R_{T_H} is the thermistor resistance at temperature T_H, and B is a constant defining the properties of the semiconductor from which the thermistor is made.

Then

$$R_T' = -\frac{B}{T^2} R_T,$$
$$R_T'' = \frac{B}{T^4}(B + 2T) R_T. \tag{7}$$

Putting expression (7) into formula (5), we get

$$\frac{rr_H}{r + r_H} = \frac{R_{T_p}}{2} \frac{B - 2T_p}{B + T_p} = R_{T_p}\gamma. \tag{8}$$

The expression for P_H is written in the form

$$P_{H1} = \frac{U^2 R_{T_1}^2 r_H}{[R_{T_1}(r + r_H) + rr_H]^2}, \quad P_{H2} = \frac{U^2 R_{T_2}^2 r_H}{[R_{T_2}(r + r_H) + rr_H]^2} \tag{9}$$

for temperatures of the medium T_1 and T_2, corresponding to the limits of the working range.

The ratio $\frac{P_{H1}}{P_{H2}}$ must be equal to the ratio P_1/P_2, which is computed or determined in the ordinary manner.

Consequently,

$$k_p = \frac{P_1}{P_2} = \frac{P_{H1}}{P_{H2}} = \frac{R_{T_1}^2\left[R_{T_2} + \frac{rr_H}{r + r_H}\right]^2}{R_{T_2}\left[R_{T_1} + \frac{rr_H}{r + r_H}\right]^2}. \tag{10}$$

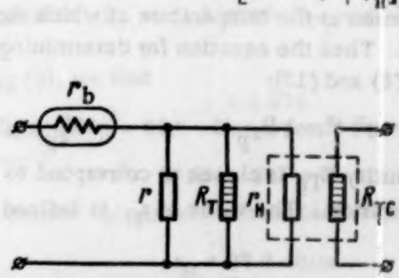

Fig. 3. Circuit for controlling the temperature of a heated thermistor with a barretter.

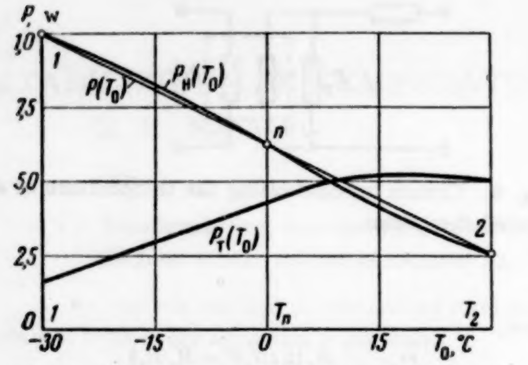

Fig. 4. Dependence of the powers P, P_H, and P_T on the temperature of the surrounding medium.

Then, according to expressions (8) and (10), we get

$$k_p = \left[\frac{R_{T_1}(R_{T_2}+\gamma R_{T_p})}{R_{T_2}(R_{T_1}+\gamma R_{T_p})}\right]^2 = \left[\frac{\varepsilon_1(\varepsilon_2+\gamma\varepsilon_p)}{\varepsilon_2(\varepsilon_1+\gamma\varepsilon_p)}\right]^2, \quad (11)$$

where

$$R_{T_K} = R_{T_H}\exp\left(\frac{B}{T_K}-\frac{B}{T_H}\right) = R_{T_H}\varepsilon_K.$$

The unknown B, which characterizes the thermistor, can be determined from (11). But since this equation is exponential, and is not analytically solved for B, then the curve $k_p(B)$ is plotted, and B, corresponding to a given k_p, is determined from it. Then the resistances r and r_H are determined. However, there is only one expression, (8), which relates these quantities. The second equation can be obtained from the condition of maximum efficiency of the circuit at one of the points in the temperature range. The circuit efficiency is defined by the expression

$$\eta = \frac{P_H}{P} = \frac{R_T^2 r_H}{(R_T+r_H)[R_T(r+r_H)+rr_H]} \quad (12)$$

where P is the power needed by the whole circuit.

The quantity r, corresponding to a maximum η, is determined from the equality $d\eta/dr_H = 0$:

$$r = \frac{r_H^2 R_T}{R_T^2 - r_H^2}. \quad (13)$$

T_p is chosen as the temperature at which the efficiency is maximum. Then the equation for determining r is obtained from (8) and (13):

$$(1-\gamma^2)r^2 - \gamma R_{T_p}(2-\gamma)r - \gamma^2 R_{T_p}^2 = 0. \quad (14)$$

The quantity R_{T_p} is chosen to correspond to existing types of thermistors. The value of r_H is defined by the expression

$$r_H = \frac{\gamma R_{T_p} r}{r-\gamma R_{T_p}}. \quad (15)$$

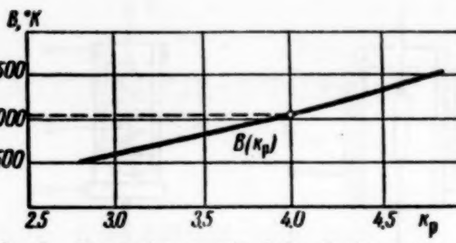

Fig. 5. Dependence $B(k_p)$ for finding the computed value of B.

Then the power supply voltage of the circuit is determined. The power, liberated in r_H at the temperature T_p, is found from the formula

$$P_{H.p} = \left[\frac{U r_H}{R_{T_p}(r+r_H)+rr_H}\right]^2 R_{T_p} \quad (16)$$

or after transformation,

$$P_{H.p} = \frac{U^2 r_H}{(r+r_H)^2}\left(\frac{1}{1+\gamma}\right)^2. \quad (17)$$

Next the supply voltage is determined:

$$U = (r+r_H)(1+\gamma)\sqrt{\frac{P_{H.p}}{r_H}}. \quad (18)$$

The next step in the calculation is to determine the maximum power dissipated in the thermistor. The power liberated in the thermistor is defined by the expression

$$P_T = \left[\frac{U r_H}{R(r+r_H)+rr_H}\right]^2 R_T. \quad (19)$$

R_T is found from the condition $dP_T/dT = 0$; it corresponds to maximum P_T:

$$R_{T_M} \doteq \frac{rr_H}{r+r_H} = \gamma R_{T_p}. \quad (20)$$

Then

$$P_{T.\max} = \frac{U^2 r_H}{4r(r+r_H)} \quad (21)$$

The temperature corresponding to $P_{T\max}$ is found from (8) and (20):

$$T_M = \frac{BT_p}{T_p \ln\gamma + B}. \quad (22)$$

If T_M proves to be outside of the working temperature range, the maximum P_T is taken as that for temperature T_2.

To conclude the computation, the maximum static error of the circuit is determined. In order to do this, the temperature at which $\Delta P = P(T_0) - P_H(T_0)$ has extreme values is found. The expression

$$2U^2 R_T R_T' r r_H^2 = -K[R_T(r+r_H)+rr_H]^3. \quad (23)$$

is found from the condition $d(\Delta P)/dT_0 = 0$.

Putting the values of R_T and R_T' from (6) and (7) into (23), we get the equation necessary to find T_{OM}:

$$2\frac{U^2 r_н^2 r}{K} = \frac{T_{ом}^2}{B}\,\frac{\left[R_{T_н}\exp\left(\frac{B}{T_н}-\frac{B}{T_{ом}}\right)(r+r_н)+rr_н\right]^3}{\left[R_{T_н}\exp\left(\frac{B}{T_н}-\frac{B}{T_{ом}}\right)\right]^2}. \quad (24)$$

Since this equation is not solved analytically for T_{OM}, one is obliged to use a graphical method. The maximum static error of the circuit is found from the formula

$$\Delta T_м = \frac{P_н(T_{ом})}{K} - (T - T_{ом}). \quad (25)$$

Let us examine the computation for a circuit with a barretter. It is characterized by constancy of the current flowing through the barretter. This current is defined by the expression

$$I = U_{12}\frac{R_т(r+r_н)+rr_н}{R_т r_н r}. \quad (26)$$

The power liberated in r_H will be

$$P_н = I^2\left[\frac{R_т}{R_т(r+r_н)+rr_н}\right]^2 r_н. \quad (27)$$

By analogy with the foregoing, an expression similar to (8) corresponds to the condition for obtaining an inflection point on the curve $P_H(T_0)$. If $r = \infty$ in the circuit, we get from (8)

$$r_н = \gamma R_{T_p}. \quad (28)$$

It is possible only in rare cases to make a circuit which will fulfill this condition. By connecting a resistance r in the circuit, one can change the total current and make it equal to the nominal current of the barretter.

The quantity B of the required thermistor is determined from the given ratio of the powers at the edges of the temperature range.

This ratio corresponds to (11) for the first circuit.

If the temperature of the heated thermistor is regulated, then r_H is the given quantity. Choosing a value for R_{Tp}, we find r:

$$r = \frac{\gamma R_{T_p} r_н}{r_н - \gamma R_{T_p}}. \quad (29)$$

The power voltage for the circuit is determined from the values of the lower limit of barretration and from the nominal current of the barretter:

$$U = I\left[\frac{R_{T_1}(r+r_н)+rr_н}{R_т rr_н}\right] + U_{L.B.} \quad (30)$$

In this expression, the thermistor resistance is taken at the lower limit of the temperature range. When the thermistor temperature is raised, the operating point on the volt-ampere characteristic of the barretter will be shifted to the middle of the barretration zone.

The power liberated in the thermistor will be

$$P_т = I^2 R_т\left[\frac{rr_н}{R_т(r+r_н)+rr_н}\right]^2. \quad (31)$$

The amount of thermistor resistance which corresponds to $P_{T\max}$ is determined by an expression similar to (20). The maximum power liberated in the thermistor will be

$$P_{T_м} = I^2\frac{rr_н}{4(r+r_н)}. \quad (32)$$

Hence, the parameters of all the elements in the circuit can be determined. In order to illustrate the method presented, the calculation of the circuit shown in Fig. 2 is presented as an example.

Example. Compute the circuit elements of the temperature regulator of a heated thermistor operating in the temperature range from $\theta_{0_1} = -30$, to $\theta_{0_2} = +30$°C. The temperature of the semiconducting element $\theta = 50$°C. The power dissipated by the heater at -30°C is $P_{H_1} = 10$ watts.

It is assumed that the best transfer between the thermistor and the medium obeys Newton's formula.

Let us find the power dissipated by the thermistor at temperatures $\theta_{0_2} = 30$°C and at $\theta_{0p} = 0$°C:

$$P_{н2} = P_{н1}\frac{\theta-\theta_{02}}{\theta-\theta_{01}} = 2.5\ \text{w},$$

$$P_{L.L.} = P_{н1}\frac{\theta-\theta_p}{\theta-\theta_{01}} = 6.25\ \text{w and } k_p = \frac{P_{н1}}{P_{н2}} = 4.$$

Table 1

B, °K	2500	3000	3500
k_p	2,83	3,92	4,8

In order to determine B corresponding to $k_p = 4$, the function $k_p(B)$ is computed from (11) in some range of change in B. The results of the calculation are presented in Table 1.

Having plotted a curve through these points (Fig. 5), we find from it that B = 3025°K, corresponding to $k_p = 4$. We will take a thermistor having $R_{20} = 20$ ohms at $T_H = 293$°K.

Then the values of the thermistor resistance at three points in the temperature range will be

$R_{T_1} = 165$ ohms $R_{T_2} = 14.2$ ohms $R_{T_p} = 42.8$ ohms

Using (8), we find

$\gamma = 0.376.$

The quantity r = 38.5 ohms is found from (14), written in the form

$r^2 - 30.5\,r - 305 = 0.$

The heater resistance is determined from (15):

$r_H = 27.8$ ohms.

The power circuit voltage is computed from (18):

U = 43.3 volts.

Table 2

θ_0, °C	−30	−15	0	15	30
ε	8.25	3.97	2.14	1.17	0.71
P, w	10.0	8.13	6.25	4.38	2.5
P_H, w	9.9	8.22	6.25	4.24	2.58
P_T, w	1.65	2.86	4.05	5.13	5.0
θ, °C	49.2	50.8	50.0	48.8	50.6

The maximum power dissipated in the thermistor will be $P_{T\max} = 5.13$ w, according to (21).

The thermistor resistance which corresponds to $P_{T\max}$ is found from (20):

$$R_{T_M} = 16.1 \text{ ohms}.$$

The temperature corresponding to R_{T_M} is found from (22): $T_M = 287$°K or 14°C.

The quantities P_H and P_T, and the temperature of the semiconducting element of the heated thermistor are determined on the basis of resistance values, the temperature dependence $R_T(T)$, and U. The results of the calculations are shown in Table 2.

The data of Table 2 confirm the correctness of the proposed methodof calculation, and show that such a circuit can be successfully used for regulating the temperature of the semiconducting element in a heated thermistor.

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USE OF HEATER RESISTANCES FOR CURRENT AND VOLTAGE STABILIZATION

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A new type of electrical stabilizer with metal heater resistance is considered and analyzed. Results of experimental investigation are given, and technical characteristics are noted.

Bridge stabilizers with nonlinear resistances – incandescent lamps, thermistors, and barretters – are widely used as low-power electric current stabilizers [1].

Bridge stabilizers have a number of advantages – simplicity, wide frequency response, and the absence of moving parts. On the other hand, such instruments have a comparatively low stabilization coefficient; furthermore, they cannot be used in circuits where the load and the supply source are commoned.

In the stabilizer developed by the author the basic advantages of bridge stabilizers are retained, and the stabilization coefficient is increased. The stabilizer is suitable for applications in circuits where the load and the supply source are commoned.

The stabilizer consists of a shunt and a metallic resistance with indirect heating. The heater resistance is represented by a platinum heater in a glass insulating tube, around which a platinum spiral is wrapped (Fig. 1). The heater and spiral are placed in an unevacuated brass container serving to protect the heater resistances from mechanical damage. The diameter of the heater is usually within the limits of 10 to 50 μ; the inner diameter of the insulating tube is approximately double that of the heater. The diameter of the spiral wire varies within the range of 5 to 30 μ. The heater resistance is attached to a socket from a type 6N8S radio tube.

The time constant τ of the heater resistances as a function of heater and spiral dimensions is within the limits 0.03 to 0.80 sec.

As indicated by the investigation, the variation of the spiral resistance R_1 with concurrent action of the heating current I and the current flowing directly through the spiral I_s is of the form

$$R_1 = R_{1_0} + cI^{\alpha} + c_1 I_s^{\beta}, \quad (1)$$

where R_{1_0} is the initial resistance of the sensing element (spiral), c and c_1 are constants depending upon the configuration of the heater resistance, α and β are constants within wide limits independent of the lay of winding and of the spiral diameter.

For platinum heaters $\alpha \approx 2.2$, where the temperature rise of the heater due to the heating current is within the

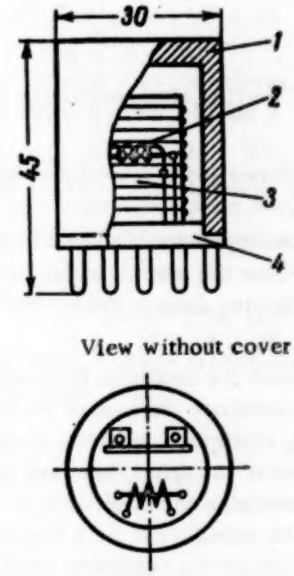

Fig. 1. Metallic resistance with indirect heater. 1) Cover; 2) heater resistance; 3) compensating resistance; 4) socket.

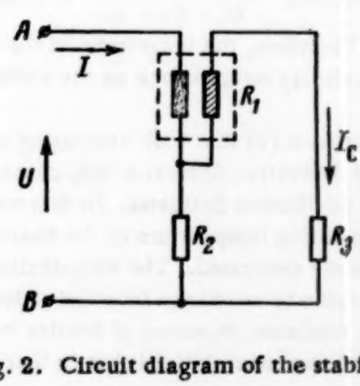

Fig. 2. Circuit diagram of the stabilizer.

limits of 40 to 400°. For heater resistances with nichrome or constantan heaters, $\alpha \approx 1.7$ under the same conditions.

With a platinum spiral, $\beta \approx 2.0$ under the condition that the temperature rise of the spiral due to the current flowing through it does not exceed 20°. With increasing temperature β does increase somewhat.

The arrangement of the stabilizer with heater resistance is shown in Fig. 2.

The relationship between current I_s in the load R_s and the heating current I is of the form

$$I_s = I\frac{R_2}{R_1 + R_2 + R_3}. \quad (2)$$

Differentiating (2) while taking (1) into account, we obtain

$$\frac{dI_s}{dI} = \frac{R_2(R_1 + R_2 + R_3) - IR_2\left(\alpha c I^{\alpha-1} + c_1\beta I_s^{\beta-1}\frac{dI_s}{dI}\right)}{(R_1 + R_2 + R_3)^2} \quad (3)$$

From that, we find, after rearrangement,

$$\frac{dI_s}{dI}\left(1 + \frac{c_1\beta I_s^{\beta}}{R_1 + R_2 + R_3}\right) = \frac{R_2\left[(R_1 + R_2 + R_3) - \alpha c I^{\alpha}\right]}{(R_1 + R_2 + R_3)^2}.$$

Equating the expression obtained to zero, we obtain the condition where the current in the load is independent of the current flowing through the heater*:

$$R_1 + R_2 + R_3 = \alpha\, I^{\alpha}. \quad (4)$$

In this manner the condition for stabilization will appear when the resistance increase of the spiral due to the heating current, multiplied α times, is equal to the sum of the resistances of the spiral, load, and shunt.

We will determine the conditions for the stabilization coefficient of the stabilizer to be a maximum. For that purpose we determine the conditions for a minimum of the second derivative d^2I_s/dI^2 in the operating sector of the stabilizer characteristic, that is, in the region where the first derivative dI_s/dI is equal to zero.

Differentiating equation (3) and setting the first derivative equal to zero, we obtain, after a number of rearrangements,

$$\frac{d^2I_s}{dI^2} = \frac{R_2(1-\alpha)}{I(R_1 + R_2 + R_2)\left[1 + \frac{c_1\beta I^{\beta}}{R_1 + R_2 + R_3}\right]}. \quad (5)$$

The expression $\frac{c_1\beta I_s^{\beta}}{R_1 + R_2 + R_3}$ is considerably less than unity. Therefore, the magnitude of the current stabilized has practically no influence on the stabilization coefficient.

It follows from (5) that with decreasing shunt resistance R_2 the second derivative decreases and, consequently, the stabilization coefficient increases. In that case, according to (4), the operating temperature of the heater resistance is simultaneously decreased. The stabilization coefficient also increases, due to resistance increase of load R_3, decrease of the nonlinear exponent of heating current α, and increase of the current stabilized; but in that case the operating temperature of the heater resistance increases, and its overload capacity decreases.

According to (4) a higher load resistance is permissible for larger values of the nonlinear exponent α; the power output of the stabilizer increases simultaneously. Therefore, a platinum wire, which has a higher nonlinear exponent than nichrome wire, is used as a heater in the stabilizer.

It was shown by the experimental investigation of a number of stabilizers devised using various metallic heater resistances that, with an increase of the heater current, the load current initially increases, and, thereafter, after reaching some maximum value, decreases. This is due to current redistribution between the load and the shunt as a result of intense resistance increase of the spiral.

The stabilization coefficient of stabilizers is, on the average, equal to 50 for a 10% change in the heating (stabilizing) current, and about 30 for a 20% change of the stabilizing current.

The resistance of the platinum heater increases significantly upon increase of heater current. Consequently, on varying the voltage applied to the heater, the resistance of the latter changes in such a fashion as to oppose a change of the heater current. As a result the stabilizer will be less sensitive to variations of terminal voltage U_{AB} than to heater-current variations.

The table shows the relationship between the load current I_s and the voltage U, transmitted to the terminals of a stabilizer with heater resistances of the platinum heater being of 50μ in diameter and a platinum spiral with wire diameter of 24 μ, with R_2 equal to 1.0 ohm, R_3 equal to 5.0 ohms, and at an ambient temperature t of 20 and 40°.

As can be seen from the table, the stabilization coefficient of a voltage stabilizer is about 70 for a 20% change in the voltage.

The load current decreases when the ambient temperature changes as a result of resistance increase of the spiral and of the heater. For current stabilization, as well as for voltage stabilization, this error can be almost fully compensated for by winding part of the shunt resistance R_2 with copper.

For current stabilizers the copper resistance is 30% of the total in R_2, and for voltage stabilizers is 50%. The compensating copper resistance, wound on a small plate, is located in the same container as the heater resistance (Fig. 1).

During ambient-temperature changes the region of stabilization is somewhat displaced. This is caused by the fact that whenever the ambient temperature increases, the condition for stabilization (4) is disturbed – resistance R_1 increases, and the heating current within the voltage stabilizer decreases slightly. The temperature error of stabilizers does not exceed 0.2% per 10°.

To protect the stabilizers from large overloads, safety fuses made of platinum wire of a diameter equal to approximately 0.8 of the heater diameter should be connected in series with the heater.

*In satisfying equation (4) the expression $\frac{c_1\beta I_s^{\beta}}{R_1 + R_2 + R_3} \ll 1$.

$t = 20°$

U, v	0.5	1.0	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
I_{c_s} ma	9.20	13.20	13.85	14.02	14.17	14.23	14.25	14.25	14.24	14.20	14.12

$t = 40°$

U, v	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
I_s ma	13.80	14.00	14.10	14.20	14.22	14.22	14.22	14.21

The stabilizers developed are suitable for stabilization of direct current and alternating current (voltage), in the frequency region of from a few cycles to several hundred kilocycles, without wave-form distortion.

As compared with thermistors, the stabilizer with heater resistance has far lower inertia, higher stability, and insignificant temperature error.

A significant advantage of the stabilizer is that it will, after an interruption in operation, reestablish, with a high degree of accuracy, previous values of stabilized current (voltage). It was shown by an investigation of three months' duration that the voltage change at the stabilizer terminals will not exceed ± 0.15% of the initial value.

The main advantages of the stabilizer are simplicity, the presence of a commoned point between the load and the supply source, and a high stabilization coefficient.

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LETTER TO THE EDITOR

Translated from Avtomatika i Telemekhanika Vol. 21, No. 4, p. 558, April, 1960

The work of the author [1] cites new equations for basic parameters of the two-point regulation process with a new time correction method. The conclusion is drawn on the basis of introduction of the concept of residual delay $\Delta t'$ so that the method of calculation approaches the method laid out in [2]. However, for practical calculation of a system with the proposed method of correction, it is expedient to express the equations for the regulation process parameters directly in terms of given values Q_{in}, Q_{out}, C, Δt, and T'_{on}. The time of starting the heat influx into the object T'_{on} is rigorously given by the correction device. In that case the equations will be as follows:

$$\Delta\vartheta'_{m(+)} = \frac{T'_{on}(Q_{in} - Q_{out}) - \Delta t Q_{out}}{C}, \tag{1}$$

$$\Delta\vartheta'_{m(-)} = \Delta\vartheta_{m(-)} = -\Delta t \frac{Q_{out}}{C}. \tag{2}$$

The full amplitude of autooscillation

$$\Delta\vartheta'_m = \Delta\vartheta'_{m(+)} + \Delta\vartheta'_{m(-)} = T'_{on} \frac{Q_{in} - Q_{out}}{c}, \tag{3}$$

$$T'_{off} = T'_{on} \frac{Q_{in} - Q_{out}}{Q_{out}} = T'_{on}(n-1), \tag{4}$$

$$T' = T'_{on} + T'_{off} = T'_{off} \frac{Q_{in}}{Q_{out}} = T'_{on} n. \tag{5}$$

Equations (1) to (5) given can be obtained from equations (1) to (5) of [1], if it is considered that

$$\Delta t' = T'_{on} - \Delta t \frac{Q_{out}}{Q_{in} - Q_{out}}.$$

In addition to the concept of degree of correction k, introduced in [1], it is also expedient to introduce a second degree of correction k_1, which represents the relationship of full oscillation amplitudes of the process without correction to the amplitudes of the process with correction, that is,

$$k_1 = \frac{\Delta\vartheta_m}{\Delta\vartheta'_m}.$$

It is apparent that

$$k_1 = \frac{\Delta t}{T'_{on}} \frac{Q_{in}}{Q_{in} - Q_{out}} = \frac{\Delta t}{T'_{on}} \frac{n}{n-1}.$$

Expressing the full period of autooscillations in the system with correction in terms of a new degree of correction k_1 and a power of influx n, we obtain

$$T' = \frac{\Delta t n^2}{(n-1)k_1}.$$

The curves of the relationship between the autooscillation period T' and a power of the influx n for $\Delta t = 1$, and at various values of the degree of correction k_1 will be of a form coinciding with the curves in Fig. 3 of [1].

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